

CLASSIFICATION OF THE AUTOMORPHISM AND ISOMETRY GROUPS OF HIGGS BUNDLE MODULI SPACES

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ABSTRACT. Let $\mathcal{M}_{n,d}$ be the moduli space of semi-stable rank n , trace-free Higgs bundles with fixed determinant of degree d on a Riemann surface of genus at least 3. We determine the following automorphism groups of $\mathcal{M}_{n,d}$: (i) the group of automorphisms as a complex analytic variety, (ii) the group of holomorphic symplectomorphisms, (iii) the group of Kähler isomorphisms, (iv) the group of automorphisms of the quaternionic structure, (v) the group of hyper-Kähler isomorphisms. When n and d are coprime we show that $\mathcal{M}_{n,d}$ admits an anti-holomorphic isomorphism if and only if the corresponding Riemann surface admits such a map. We then use this to determine the isometry group of $\mathcal{M}_{n,d}$.

1. INTRODUCTION

Introduced by Hitchin in his groundbreaking 1987 paper [14], the moduli spaces of semi-stable Higgs bundles on a compact Riemann surface Σ are complex algebraic varieties possessing a remarkable wealth of geometric structures. Most notably these moduli spaces are algebraically completely integrable systems and their smooth points are hyper-Kähler manifolds. As with their Kähler counterparts, the moduli spaces of semi-stable bundles on Σ , the study of Higgs bundle moduli spaces has proven to be an extremely fruitful pursuit. Through the non-abelian Hodge theory developed by Corlette, Donaldson, Hitchin, Simpson and others [8, 9, 14, 27], the Higgs bundle moduli spaces can be identified with the character varieties of representations of the fundamental group of Σ into a complex reductive Lie group. In another direction, it has been proposed that the geometric Langlands correspondence should be understood as a mirror symmetry of Higgs bundle moduli spaces for Langlands dual groups [13, 18]. Consequently there is considerable impetus for the study of Higgs bundle moduli spaces.

Given a holomorphic line bundle L_0 on Σ , we let \mathcal{M}_{n,L_0} denote the moduli space of rank n , trace-free Higgs bundles with determinant L_0 . In this paper we determine the automorphism group of the moduli space \mathcal{M}_{n,L_0} as a complex analytic variety, provided the genus g of Σ is at least 3. We also determine several related symmetry groups of \mathcal{M}_{n,L_0} ; the group of holomorphic symplectomorphisms, the group of Kähler isomorphisms, the group of automorphisms of the quaternionic structure and the group of hyper-Kähler isomorphisms. In the case that n and the degree of L_0 are coprime, we also determine the anti-holomorphic automorphisms

Date: November 11, 2014.

2010 *Mathematics Subject Classification.* Primary 14H60 53C07; Secondary 14H70, 53C26.

This work is supported by the Australian Research Council Discovery Project DP110103745.

of \mathcal{M}_{n,L_0} and the group of isometries of \mathcal{M}_{n,L_0} .

To put our results in context, we recall the classification of the automorphism group of SU_{n,L_0} , the moduli space of rank n stable bundles E with fixed determinant $\det(E) \cong L_0$:

Theorem 1.1 (Kouvidakis-Pantev [19], Hwang-Ramanan [17]). *For $g \geq 3$ the automorphism group $\text{Aut}(SU_{n,L_0})$ of SU_{n,L_0} is generated by the following automorphisms:*

- (1) $E \mapsto \sigma^* E \otimes L$, where $\sigma : \Sigma \rightarrow \Sigma$ is an automorphism and L is a holomorphic line bundle with $L^n \cong L_0 \otimes \sigma^* L_0^*$,
- (2) $E \mapsto \sigma^* E^* \otimes L$, where $\sigma : \Sigma \rightarrow \Sigma$ is again an automorphism and L is a holomorphic line bundle with $L^n \cong L_0 \otimes \sigma^* L_0$.

Note that automorphisms of type (2) can only occur if n divides $2d$, where $d = \deg(L_0)$ is the degree of L_0 .

Evidently the automorphism group of the moduli space SU_{n,L_0} is finite, as long as $g \geq 3$. In contrast we will see that the Higgs bundle moduli space \mathcal{M}_{n,L_0} has a much richer automorphism group. Not only is this group infinite, it is in a sense infinite dimensional. To describe this group we consider three ways of producing automorphisms of \mathcal{M}_{n,L_0} :

1. The \mathbb{C}^* -action. There is a natural \mathbb{C}^* -action on \mathcal{M}_{n,L_0} of the form $(E, \Phi) \mapsto (E, \lambda\Phi)$, for $\lambda \in \mathbb{C}^*$. This action has proven to be an indispensable tool in the study of the topology of Higgs bundle moduli spaces. Notably, the circle subgroup $U(1) \subset \mathbb{C}^*$ has an associated moment map which has been used extensively to study \mathcal{M}_{n,L_0} through Morse-theoretic techniques [14, 11, 7].

2. Automorphisms of SU_{n,L_0} . Let SU_{n,L_0}^{sm} denote the smooth points of SU_{n,L_0} . The second class of automorphisms we wish to consider exploits the natural inclusion $T^*SU_{n,L_0}^{\text{sm}} \subset \mathcal{M}_{n,L_0}$ of the cotangent bundle of SU_{n,L_0}^{sm} as a dense open subset of \mathcal{M}_{n,L_0} . By differentiation, an automorphism $\phi : SU_{n,L_0}^{\text{sm}} \rightarrow SU_{n,L_0}^{\text{sm}}$ defines an automorphism of the cotangent bundle $\phi_* = (\phi^*)^{-1} : T^*SU_{n,L_0}^{\text{sm}} \rightarrow T^*SU_{n,L_0}^{\text{sm}}$. Clearly the automorphisms described in Theorem 1.1 extend to the whole moduli space \mathcal{M}_{n,L_0} . In this way $\text{Aut}(SU_{n,L_0})$ can be identified with a subgroup of $\text{Aut}(\mathcal{M}_{n,L_0})$.

3. Vertical flows of the Hitchin system. To describe these automorphisms we first recall the *Hitchin system* [15]. The Higgs bundle moduli space carries a naturally defined holomorphic symplectic form Ω_I extending the canonical symplectic structure on $T^*SU_{n,L_0}^{\text{sm}}$. As we shall recall in the paper, \mathcal{M}_{n,L_0} is an algebraically completely integrable system. Specifically, there exists holomorphic Poisson commuting functions h_1, \dots, h_m on \mathcal{M}_{n,L_0} , where $m = \frac{1}{2}\dim(\mathcal{M}_{n,L_0})$, such that dh_1, \dots, dh_m are generically independent and the generic fibre of the Hitchin map

$$h = (h_1, \dots, h_m) : \mathcal{M}_{n,L_0} \rightarrow \mathbb{C}^m$$

is an abelian variety. Let $\mathcal{A} = \mathbb{C}^m$ denote the base of the Hitchin system. To each holomorphic function f on \mathcal{A} there is an associated Hamiltonian vector field X_f . The properness of the Hitchin map ensures that X_f can be integrated to a

symplectomorphism e^{X_f} of $\mathcal{M}_{n,L_0}^{\text{sm}}$, the smooth locus of $\mathcal{M}_{n,d}$. We will see that e^{X_μ} actually extends to an automorphism of the full moduli space. More generally if μ is a holomorphic 1-form on \mathcal{A} we have an associated vector field X_μ determined by the relation $i_{X_\mu}\Omega_I = h^*(\mu)$. Such a vector field can likewise be integrated to an automorphism e^{X_μ} of $\mathcal{M}_{n,L_0}^{\text{sm}}$, which is not a symplectomorphism unless μ is exact. Similarly we find that e^{X_μ} extends to an automorphism of the full moduli space. We also show that the set of all automorphisms of the form e^{X_μ} gives an abelian subgroup of $\text{Aut}(\mathcal{M}_{n,L_0})$, isomorphic to the set of holomorphic 1-forms on \mathcal{A} under addition. We denote this subgroup by $\text{Vert}_0(\mathcal{M}_{n,L_0}) \subset \text{Aut}(\mathcal{M}_{n,L_0})$. While Hamiltonian flows of the Hitchin system have been considered by several authors (e.g., [15, 16, 4]), the larger group $\text{Vert}_0(\mathcal{M}_{n,L_0})$ seems to have been given little consideration. The group $\text{Vert}_0(\mathcal{M}_{n,L_0})$ bears a close relationship to a construction central to Ngô's proof of the fundamental lemma. Notice that on the non-singular fibres of $h : \mathcal{M}_{n,L_0} \rightarrow \mathcal{A}$, which are abelian varieties, the group $\text{Vert}_0(\mathcal{M}_{n,L_0})$ acts by translation on each fibre. This is similar to the construction in [23, 24] of a group scheme $P^{\text{ell}} \rightarrow \mathcal{A}^{\text{ell}}$, over the so-called elliptic locus $\mathcal{A}^{\text{ell}} \subset \mathcal{A}$ which carries a fibrewise action $P^{\text{ell}} \times_{\mathcal{A}^{\text{ell}}} \mathcal{M}_{n,L_0}^{\text{ell}} \rightarrow \mathcal{M}_{n,L_0}^{\text{ell}}$. On the non-singular fibres this action is again by translations.

Our main theorem is that the above three classes of automorphisms generate the automorphism group. More precisely we have:

Theorem 1.2. *Let Σ have genus $g \geq 3$. The subgroup $\text{Vert}_0(\mathcal{M}_{n,L_0}) \subset \text{Aut}(\mathcal{M}_{n,L_0})$ is normal. Moreover we have an isomorphism:*

$$\text{Aut}(\mathcal{M}_{n,L_0}) = (\mathbb{C}^* \times \text{Aut}(SU_{n,L_0})) \ltimes \text{Vert}_0(\mathcal{M}_{n,L_0}).$$

Using an infinite dimensional hyper-Kähler quotient construction [14], Hitchin showed that on the smooth points $\mathcal{M}_{n,L_0}^{\text{sm}}$ of \mathcal{M}_{n,L_0} , there is a natural hyper-Kähler structure. To be specific, this consists of integrable complex structures I, J, K satisfying the quaternionic relation $IJ = K$ and a Riemannian metric g which is Kähler with respect to I, J and K . In terms of the associated Kähler forms $\omega_I, \omega_J, \omega_K$, the holomorphic symplectic form Ω_I is given by $\Omega_I = \omega_J + i\omega_K$. In addition to the automorphisms of \mathcal{M}_{n,L_0} as a complex analytic variety, we also consider various subgroups of $\text{Aut}(\mathcal{M}_{n,L_0})$ preserving different parts of the hyper-Kähler structure:

Definition 1.3. We define the following subgroups of $\text{Aut}(\mathcal{M}_{n,L_0})$:

- $\text{Aut}_{\text{Sympl}}(\mathcal{M}_{n,L_0}) = \{\phi \in \text{Aut}(\mathcal{M}_{n,L_0}) \mid \phi|_{\mathcal{M}_{n,L_0}^{\text{sm}}} \text{ preserves } \Omega_I\}$, the group of holomorphic symplectomorphisms of \mathcal{M}_{n,L_0} .
- $\text{Aut}_{\text{Isom}}(\mathcal{M}_{n,L_0}) = \{\phi \in \text{Aut}(\mathcal{M}_{n,L_0}) \mid \phi|_{\mathcal{M}_{n,L_0}^{\text{sm}}} \text{ preserves } g\}$, the group of holomorphic isometries of \mathcal{M}_{n,L_0} .
- $\text{Aut}_Q(\mathcal{M}_{n,L_0}) = \{\phi \in \text{Aut}(\mathcal{M}_{n,L_0}) \mid \phi|_{\mathcal{M}_{n,L_0}^{\text{sm}}} \text{ preserves } J\}$, the group of quaternionic isomorphisms of \mathcal{M}_{n,L_0} .
- $\text{Aut}_{HK}(\mathcal{M}_{n,L_0}) = \{\phi \in \text{Aut}(\mathcal{M}_{n,L_0}) \mid \phi|_{\mathcal{M}_{n,L_0}^{\text{sm}}} \text{ preserves } g, J\}$, the group of hyper-Kähler isomorphisms of \mathcal{M}_{n,L_0} .

Our second main theorem is a complete description of these subgroups:

Theorem 1.4. *Under the isomorphism $\text{Aut}(\mathcal{M}_{n,L_0}) \cong (\mathbb{C}^* \times \text{Aut}(SU_{n,L_0})) \ltimes \text{Vert}_0(\mathcal{M}_{n,L_0})$, the subgroups given in Definition 1.3 are as follows:*

- (1) $Aut_{Sympl}(\mathcal{M}_{n,L_0}) = (\{1\} \times Aut(SU_{n,L_0})) \ltimes Ham(\mathcal{M}_{n,d})$,
- (2) $Aut_{Isom}(\mathcal{M}_{n,L_0}) = (U(1) \times Aut(SU_{n,L_0}))$,
- (3) $Aut_Q(\mathcal{M}_{n,L_0}) = (\mathbb{R}_+ \times Aut(SU_{n,L_0}))$,
- (4) $Aut_{HK}(\mathcal{M}_{n,L_0}) = (\{1\} \times Aut(SU_{n,L_0}))$,

where $U(1) \subset \mathbb{C}^*$ is the subgroup of unit complex numbers and $\mathbb{R}_+ \subset \mathbb{C}^*$ is the subgroup of positive real numbers.

In this theorem, $Ham(\mathcal{M}_{n,L_0})$ is the subgroup of $Vert_0(\mathcal{M}_{n,L_0})$ consisting of the Hamiltonian flows associated to holomorphic functions on \mathcal{A} .

We consider two further kinds of symmetries associated to \mathcal{M}_{n,L_0} . For this we will assume that $d = \deg(L_0)$ is coprime to n , so that \mathcal{M}_{n,L_0} is a smooth hyper-Kähler manifold. By an *anti-automorphism* of a complex manifold (X, I) , we mean a diffeomorphism $\phi : X \rightarrow X$ such that $\phi_* \circ I = -I \circ \phi_*$.

Theorem 1.5. *Suppose that n and d are coprime. Then $\mathcal{M}_{n,d}$ admits an anti-automorphism if and only if Σ admits an anti-automorphism.*

Using this, we are able to determine the full isometry group of \mathcal{M}_{n,L_0} :

Theorem 1.6. *Suppose n and $d = \deg(L_0)$ are coprime.*

- *If Σ does not admit an anti-automorphism, then every isometry of \mathcal{M}_{n,L_0} preserves I . Therefore $Isom(\mathcal{M}_{n,L_0}) = Aut_{Isom}(\mathcal{M}_{n,L_0}) \cong (U(1) \times Aut(SU_{n,L_0}))$.*
- *If Σ admits an anti-automorphism then the subgroup of isometries of $\mathcal{M}_{n,d}$ preserving I has index 2 in the isometry group of \mathcal{M}_{n,L_0} .*

In the case that Σ admits an anti-automorphism we can say more precisely what the isometry group $Isom(\mathcal{M}_{n,L_0})$ is. We leave the details to Section 7.2.

The plan of the paper is as follows. Section 2 is a review of Higgs bundle moduli spaces, the Hitchin system and spectral curves. In Section 3 we study in depth the Hamiltonian flows of the Hitchin system on non-singular fibres in §3.1 and then on generic singular fibres in §3.2. This will allow us to deduce that the Hitchin map is a submersion on a suitably large open subset. In Section 4, we give a classification of the holomorphic vector fields. To do this, we consider in §4.1 the Kodaira-Spencer mapping associated to the non-singular fibres of the Hitchin system. This allows us in §4.2 to determine the holomorphic vector fields on the complement of the singular fibres. Using the results of Section 3 we can then deduce which holomorphic vector fields extend to the entire smooth locus of the moduli space. Based on our study of flows of the Hitchin system, we are further able to determine which holomorphic vector fields integrate to automorphisms of the smooth locus and by inspection we see that these extend as automorphisms to the whole moduli space.

In Section 5, we prove our main result, Theorem 1.2. The key idea is that if ϕ is an automorphism of \mathcal{M}_{n,L_0} , then we can find a flow e^{X_ν} along the fibres of the Hitchin system such that the composition $e^{X_\nu} \circ \phi$ commutes with the \mathbb{C}^* -action. We are able to accomplish this by using our classification of the holomorphic vector fields on $\mathcal{M}_{n,L_0}^{\text{sm}}$. This reduces the problem to the classification of automorphisms commuting with the \mathbb{C}^* -action. We are then able to reduce this to the problem of determining automorphisms of the cotangent bundle $T^*SU_{n,d}^{\text{sm}}$ which are linear in the fibres. Using once again our classification of holomorphic vector fields on the moduli space,

we are able to determine the group of all such automorphisms and the main theorem follows. In Section 6 we give the proof of Theorem 1.4. Our main strategy is to examine how automorphisms which preserve some of the geometry of \mathcal{M}_{n,L_0} act on the rest of the geometry. Our classification of holomorphic vector fields on $\mathcal{M}_{n,L_0}^{\text{sm}}$ turns out to be crucial to this approach. Finally in Section 7 we consider anti-automorphisms and isometries of the moduli space under the assumption of coprime rank and degree. In §7.1 we prove Theorem 1.5 as a consequence of the Torelli theorem for Higgs bundle moduli spaces [5]. In §7.2 we prove Theorem 1.6 and consequently a classification of the isometry groups of these moduli spaces.

2. HIGGS BUNDLES AND THE HITCHIN SYSTEM

2.1. Higgs bundle moduli spaces. Let Σ be a compact Riemann surface of genus $g > 1$ and let K denote the canonical bundle of Σ . A *Higgs bundle* of rank n , degree d on Σ is a pair (E, Φ) , where E is a holomorphic vector bundle of rank n , degree d and Φ is a holomorphic section of $\text{End}(E) \otimes K$, called the *Higgs field*. It is often convenient to think of E as consisting of an underlying \mathcal{C}^∞ vector bundle, which we also denote by E , together with a choice of holomorphic structure on E . The holomorphic structure on E is specified by giving a $\bar{\partial}$ -operator, $\bar{\partial}_E : \Omega^0(\Sigma, E) \rightarrow \Omega^{0,1}(\Sigma, E)$ and the requirement that Φ is holomorphic reads $\bar{\partial}_E \Phi = 0$. As such we will often denote Higgs bundles as pairs $(\bar{\partial}_E, \Phi)$.

The *slope* $\mu(E)$ of a holomorphic vector bundle E on Σ is defined as the quotient $\mu(E) = \deg(E)/\text{rank}(E)$. Similarly the slope $\mu(E, \Phi)$ of a Higgs bundle (E, Φ) is simply the slope of E , $\mu(E, \Phi) = \mu(E)$. Recall that a Higgs bundle (E, Φ) is said to be *semi-stable* if for all proper Φ -invariant subbundles $F \subset E$, we have $\mu(F) \leq \mu(E)$. We say that (E, Φ) is *stable* if this inequality is strict for all such F . Any semi-stable Higgs bundle (E, Φ) has a Jordan-Hölder filtration, that is, a sequence $0 = E_1 \subset \cdots \subset E_l = E$ of Φ -invariant subbundles for which $\mu(E_i/E_{i-1}) = \mu(E)$ and the induced Higgs bundles $(E_i/E_{i-1}, \Phi_i)$ are stable. The associated graded Higgs bundle $\text{gr}(E, \Phi) = \bigoplus_i (E_i/E_{i-1}, \Phi_i)$ is determined up to isomorphism from (E, Φ) [22]. Two semi-stable Higgs bundles are said to be *S-equivalent* if their associated graded Higgs bundles are isomorphic. In particular, two stable Higgs bundles are S-equivalent if and only if they are isomorphic.

We say that a Higgs bundle (E, Φ) is *trace-free* if Φ , viewed as a K -valued endomorphism of E is trace-free. Suppose that L_0 is a holomorphic line bundle of degree d . We say that (E, Φ) has determinant L_0 if $\det(E) \cong L_0$.

Definition 2.1. We let \mathcal{M}_{n,L_0} denote the moduli space of S-equivalence classes of rank n , degree d Higgs bundles, which are trace-free and have determinant L_0 . The space \mathcal{M}_{n,L_0} is a quasi-projective algebraic variety over \mathbb{C} and has an open subvariety \mathcal{M}_{n,L_0}^s , the moduli space of stable rank n , degree d Higgs bundle, trace-free with determinant L_0 [22]. The dimension of $\mathcal{M}_{n,d}$ is $2(n^2 - 1)(g - 1)$.

Remark 2.2. While \mathcal{M}_{n,L_0} has an algebraic structure, we will study the automorphisms of \mathcal{M}_{n,L_0} within the category of complex analytic varieties. This is an important point to bear this in mind, as \mathcal{M}_{n,L_0} is non-compact.

Remark 2.3. For any line bundle L , the tensor product $(E, \Phi) \mapsto (E \otimes L, \Phi \otimes \text{Id})$ defines an isomorphism $\otimes L : \mathcal{M}_{n,L_0} \rightarrow \mathcal{M}_{n,L_0 \otimes L^n}$. It follows that if L_0, L'_0 have the

same degree modulo n , there is an isomorphism $\mathcal{M}_{n,L_0} \cong \mathcal{M}_{n,L'_0}$. As an algebraic variety, \mathcal{M}_{n,L_0} only depends on L_0 through the degree $d = \deg(L_0)$ and as such we will often use $\mathcal{M}_{n,d}$ to denote \mathcal{M}_{n,L_0} , where L_0 is any holomorphic line bundle of degree d .

Let $\mathcal{M}_{n,d}^{\text{sm}} \subseteq \mathcal{M}_{n,d}$ denote the locus of smooth points of $\mathcal{M}_{n,d}$ and note that stable points are smooth, so $\mathcal{M}_{n,d}^s \subseteq \mathcal{M}_{n,d}^{\text{sm}}$. We also note that when n and d are coprime, every semistable Higgs bundle is stable so that $\mathcal{M}_{n,d}^s = \mathcal{M}_{n,d}^{\text{sm}} = \mathcal{M}_{n,d}$ and the moduli space is a complex manifold. For a smooth point $m \in \mathcal{M}_{n,d}^{\text{sm}}$ represented by a Higgs bundle pair (E, Φ) , the tangent space $T_m \mathcal{M}_{n,d}^{\text{sm}}$ may be described as follows. Let $\text{End}_0(E) \subset \text{End}(E)$ denotes the subbundle of trace-free endomorphisms of E . An infinitesimal deformation of (E, Φ) is represented by a pair $(\dot{A}, \dot{\Phi}) \in \Omega^{0,1}(\Sigma, \text{End}_0(E)) \oplus \Omega^{1,0}(\Sigma, \text{End}_0(E))$, where \dot{A} is a deformation of the holomorphic structure $\bar{\partial}_E$ and $\dot{\Phi}$ is a deformation of Φ . Differentiating the condition $\bar{\partial}_E \Phi = 0$, we see that such pairs $(\dot{A}, \dot{\Phi})$ must satisfy $\bar{\partial}_E \dot{\Phi} + [\dot{A}, \Phi] = 0$. Further, a pair $(\dot{A}, \dot{\Phi})$ gives a trivial deformation whenever $(\dot{A}, \dot{\Phi}) = (\bar{\partial}_E \psi, [\Phi, \psi])$, for some $\psi \in \Omega^0(\Sigma, \text{End}_0(E))$. It can then be shown that this gives a natural identification of $T_m \mathcal{M}_{n,d}^{\text{sm}}$ with the space of pairs $(\dot{A}, \dot{\Phi})$ such that $\bar{\partial}_E \dot{\Phi} + [\dot{A}, \Phi] = 0$, modulo pairs of the form $(\dot{A}, \dot{\Phi}) = (\bar{\partial}_E \psi, [\Phi, \psi])$. This is precisely the degree 1 hypercohomology of the complex

$$\text{End}_0(E) \xrightarrow{[\Phi, \cdot]} \text{End}_0(E) \otimes K.$$

The smooth locus $\mathcal{M}_{n,d}^{\text{sm}}$ is a complex manifold. We let I denote the complex structure on $\mathcal{M}_{n,d}^{\text{sm}}$. In terms of deformations $(\dot{A}, \dot{\Phi})$, the complex structure is simply the natural complex structure $I(\dot{A}, \dot{\Phi}) = (i\dot{A}, i\dot{\Phi})$. Furthermore $\mathcal{M}_{n,d}^{\text{sm}}$ carries a natural holomorphic symplectic form Ω_I , which may be defined as follows:

$$(2.1) \quad \Omega_I((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_{\Sigma} \text{Tr}(\dot{A}_1 \dot{\Phi}_2 - \dot{A}_2 \dot{\Phi}_1),$$

where Tr denotes the trace of an endomorphism. The 2-form Ω_I is closed and defines a holomorphic symplectic structure on $\mathcal{M}_{n,d}^{\text{sm}}$. One way to see this is to construct Ω_I by means of an infinite dimensional symplectic quotient [14, 15] or indeed a hyper-Kähler quotient. We will consider the hyper-Kähler structure of $\mathcal{M}_{n,d}^{\text{sm}}$ in Section 6. For now we only need to make use of the holomorphic symplectic structure (I, Ω_I) .

A Higgs bundle $(E, 0)$ with zero Higgs field is simply a holomorphic vector bundle E . For such Higgs bundles (semi)-stability reduces to the usual definition of (semi)-stability of holomorphic bundles. There is likewise a moduli space \mathcal{SU}_{n,L_0} of S -equivalence classes of rank n , degree d semi-stable bundles with determinant L_0 . As with Higgs bundles, this space only depends on L_0 through the degree mod n , so we will also denote the moduli space by $\mathcal{SU}_{n,d}$. This space is a complex projective variety of dimension $(n^2 - 1)(g - 1)$. We let $\mathcal{SU}_{n,d}^{\text{sm}} \subseteq \mathcal{SU}_{n,d}$ denote the smooth locus. Then for a point $[E] \in \mathcal{SU}_{n,d}^{\text{sm}}$ the tangent space is $T_{[E]} \mathcal{SU}_{n,d}^{\text{sm}} \cong H^1(\Sigma, \text{End}_0(E))$. By Serre duality, the cotangent space is $T_{[E]}^* \mathcal{SU}_{n,d}^{\text{sm}} \cong H^0(\Sigma, \text{End}_0(E) \otimes K)$. Thus a cotangent vector $\Phi \in T_{[E]}^* \mathcal{SU}_{n,d}^{\text{sm}}$ can be thought of as a Higgs field on E and one sees that (E, Φ) defines a point in $\mathcal{M}_{n,d}^{\text{sm}}$. In this way we obtain a natural open inclusion

$T^*\mathcal{SU}_{n,d}^{\text{sm}} \subset \mathcal{M}_{n,d}^{\text{sm}}$. Note also that the symplectic form Ω_I on $\mathcal{M}_{n,d}^{\text{sm}}$ restricts to the canonical symplectic form on $T^*\mathcal{SU}_{n,d}^{\text{sm}}$.

2.2. Spectral curves and the Hitchin system. Identify the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ with the space of trace-free endomorphisms of \mathbb{C}^n . Any $A \in \mathfrak{sl}(n, \mathbb{C})$ has a characteristic polynomial:

$$\det(\lambda - A) = \lambda^n + a_2(A)\lambda^{n-2} + \cdots + a_n(A).$$

The coefficients a_2, \dots, a_n are generators for the ring of polynomials on $\mathfrak{sl}(n, \mathbb{C})$ which are invariant under the adjoint action of $SL(n, \mathbb{C})$. Note that a_j is homogeneous of degree j and we can alternatively define a_j as:

$$(2.2) \quad a_j(A) = (-1)^j \text{Tr}_{\wedge^j \mathbb{C}^n}(A),$$

where for any $\mathfrak{sl}(n, \mathbb{C})$ -representation R , we let $\text{Tr}_R(A)$ denote the trace of A in the representation R . When $R = \mathbb{C}^n$ is the standard representation we write this simply as Tr . There are many other generating sets for the ring of invariant polynomials, we mention just two other sets $\{b_j\}_{j=2}^n, \{h_j\}_{j=2}^n$ which will be important to us:

$$(2.3) \quad b_j(A) = \frac{1}{j} \text{Tr}(A^j),$$

$$(2.4) \quad h_j(A) = \text{Tr}_{S^j(\mathbb{C}^n)}(A).$$

Note that b_j and h_j are also homogeneous of degree j . The invariants a_j, b_j, h_j are defined for every integer $j \geq 1$ by Equations (2.2)-(2.4). Since A is trace-free we have $a_1 = b_1 = h_1 = 0$. For convenience we also set $a_0 = h_0 = 1$ but leave b_0 undefined. The generating sets $\{a_j\}_{j=2}^n, \{b_j\}_{j=2}^n, \{h_j\}_{j=2}^n$ are related by the following versions of Newton's identities [20], valid for any $k \geq 1$:

$$(2.5) \quad \sum_{j=1}^k j b_j a_{k-j} = -k a_k, \quad \sum_{j=1}^k j b_j h_{k-j} = k h_k, \quad \sum_{j=0}^k h_j a_{k-j} = 0.$$

Suppose that (E, Φ) is a rank n trace-free Higgs bundle. If f_j is any degree j invariant polynomial on $\mathfrak{sl}(n, \mathbb{C})$, then we can apply f_j to Φ to obtain a holomorphic section $f_j(\Phi)$ of K^j . Invariance of f_j further ensures that $f_j(\Phi)$ depends only on the isomorphism class of (E, Φ) . Let f_2, \dots, f_n be a set of generators for the ring of invariant polynomials, where f_j is homogeneous of j . This defines a holomorphic map $h : \mathcal{M}_{n,d} \rightarrow \mathcal{A}$ into the affine space

$$\mathcal{A} = \bigoplus_{j=2}^n H^0(\Sigma, K^j),$$

by evaluating the polynomials f_j on the Higgs field:

$$h(E, \Phi) = (f_2(\Phi), f_3(\Phi), \dots, f_n(\Phi)).$$

The map h is known as the *Hitchin map* and $h : \mathcal{M}_{n,d} \rightarrow \mathcal{A}$ is referred to as the *Hitchin fibration* or *Hitchin system*. Given two different choices of generators for the ring of invariant polynomials $\{f_j\}, \{f'_j\}$, the corresponding Hitchin maps h, h' are related by an isomorphism of \mathcal{A} . In this sense the Hitchin map is essentially independent of the choice of generators. However, it will be convenient to take the Hitchin map with respect to b_2, \dots, b_n in (2.3). Therefore we define the Hitchin map $h : \mathcal{M}_{n,d} \rightarrow \mathcal{A}$ from now on to be given by

$$h(E, \Phi) = (b_2(\Phi), b_3(\Phi), \dots, b_n(\Phi)) = \left(\frac{1}{2}\text{Tr}(\Phi^2), \frac{1}{3}\text{Tr}(\Phi^3), \dots, \frac{1}{n}\text{Tr}(\Phi^n)\right).$$

When no confusion is likely to arise we will denote $a_j(\Phi), b_j(\Phi), h_j(\Phi) \in H^0(\Sigma, K^j)$ simply as a_j, b_j, h_j . Newton's identities (2.5) allow us to express the $a_j(\Phi)$ and $h_j(\Phi)$ in terms of the $b_j(\Phi)$. In particular, there is an automorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ of the affine variety \mathcal{A} such that for any (E, Φ) we have $\alpha(b_2(\Phi), \dots, b_n(\Phi)) = (a_2(\Phi), \dots, a_n(\Phi))$. For a Higgs bundle $(E, \Phi) \in \mathcal{M}_{n,d}$ we will often use to $b = (b_2, \dots, b_n) \in \mathcal{A}$ to denote $h(E, \Phi)$ and $a = (a_2, \dots, a_n) \in \mathcal{A}$ will denote $\alpha(b)$.

The Hitchin fibration is an *algebraically completely integrable system* [15]. Recall that this means $h : \mathcal{M}_{n,d} \rightarrow \mathcal{A}$ is a holomorphic Lagrangian fibration with respect to Ω_I , that the generic fibre of h is an open set in an abelian variety and that the Hamiltonian vector fields $X_{f_1}, X_{f_2}, \dots, X_{f_m}$ are linear on the fibres, where f_1, \dots, f_m are coordinate functions on \mathcal{A} . As we will recall, in the case of the Hitchin fibration, the generic fibres are actually abelian varieties. This guarantees that the vector fields X_{f_1}, \dots, X_{f_m} are linear on the fibres, since every global holomorphic vector field on a complex torus is linear.

The fibres of the Hitchin fibration can be described using the notion of spectral curves. For this, suppose we are given $b = (b_2, \dots, b_n) \in \mathcal{A}$ and set $a = \alpha(b) = (a_2, \dots, a_n) \in \mathcal{A}$, so $a_j \in H^0(\Sigma, K^j)$. Let $\pi : K \rightarrow \Sigma$ denote the projection from the total space of K to Σ and let λ denote the tautological section of $\pi^*(K)$. Consider the section $s_b \in H^0(K, \pi^*(K^n))$ given by

$$(2.6) \quad s_b = \lambda^n + \pi^*(a_2)\lambda^{n-2} + \dots + \pi^*(a_n).$$

The zero locus $S_b \subset K$ of s_b is called the *spectral curve* associated to b . In general S_b can be singular, however, Bertini's theorem implies that S_b is smooth for generic $b \in \mathcal{A}$. Let us define the *discriminant divisor* $\mathcal{D} \subset \mathcal{A}$ as

$$\mathcal{D} = \{b \in \mathcal{A} \mid S_b \text{ is not smooth}\}.$$

It can be shown that \mathcal{D} is an irreducible divisor in \mathcal{A} [19, Corollary 1.5]. Any Higgs bundle $(E, \Phi) \in \mathcal{M}_{n,d}$ defines a spectral curve given by the characteristic equation of Φ :

$$\det(\lambda - \pi^*(\Phi)) = 0.$$

Note that $\det(\lambda - \pi^*(\Phi)) = s_b$, where $b = h(E, \Phi)$ and s_b is given by Equation (2.6). So the spectral curve $\det(\lambda - \pi^*(\Phi)) = 0$ associated to (E, Φ) is precisely the spectral curve associated to $b = h(E, \Phi) \in \mathcal{A}$.

Let $\mathcal{A}^{\text{reg}} = \mathcal{A} - \mathcal{D}$ denote the complement of the discriminant divisor. We say that $b \in \mathcal{A}$ is *regular* if $b \in \mathcal{A}^{\text{reg}}$. Thus b is regular precisely if the spectral curve S_b is smooth. Similarly we let $\mathcal{M}_{n,d}^{\text{reg}} = h^{-1}(\mathcal{A}^{\text{reg}})$ be the space of Higgs bundles with smooth spectral curve. Then $\mathcal{M}_{n,d}^{\text{reg}} \subset \mathcal{M}_{n,d}$ is a dense open subset. Moreover one can show that every element of $\mathcal{M}_{n,d}^{\text{reg}}$ is stable, so that $\mathcal{M}_{n,d}^{\text{reg}} \subseteq \mathcal{M}_{n,d}^{\text{sm}}$ [3].

Let $b \in \mathcal{A}^{\text{reg}}$ and let $\pi : S_b \rightarrow \Sigma$ be the associated spectral curve. To simplify notation we will denote S_b simply as S when no confusion is likely to occur. Let K_S denote the canonical bundle of S . By the adjunction formula we have $K_S \cong \pi^*(K^n)$. Set $\tilde{d} = d + n(n-1)(g-1)$ and let $\text{Jac}_{\tilde{d}}(S)$ denote the space of degree \tilde{d} line bundles on S . If $L \in \text{Jac}_{\tilde{d}}(S)$ we have by Grothendieck-Riemann-Roch that $E = \pi_*(L)$ is a rank n , degree d holomorphic vector bundle on Σ . The tautological section λ

may be viewed as a map $\lambda : L \rightarrow L \otimes \pi^* K$, which then pushes down to a map $\Phi : E \rightarrow E \otimes K$. In this way we have constructed from L a Higgs bundle pair (E, Φ) . One can then show that $\det(\lambda - \pi^*(\Phi)) = \lambda^n + a_2 \lambda^{n-2} + \dots + a_n$ [3], so S is the spectral curve associated to (E, Φ) . Define the Prym variety $\text{Prym}_{\bar{d}}(S, \Sigma)$ as follows:

$$\text{Prym}_{\bar{d}}(S, \Sigma) = \{L \in \text{Jac}_{\bar{d}}(S) \mid \det(\pi_* L) \cong L_0\}.$$

Therefore if $L \in \text{Prym}_{\bar{d}}(S, \Sigma)$, the associated Higgs bundle (E, Φ) is trace-free with determinant L_0 . Conversely, any $(E, \Phi) \in \mathcal{M}_{n,d}$ with associated spectral curve S is obtained in this way from some $L \in \text{Prym}_{\bar{d}}(S, \Sigma)$ [15, 3]. This shows that the fibre $h^{-1}(b)$ of the Hitchin system over b is the abelian variety $\text{Prym}_{\bar{d}}(S, \Sigma)$. Let $Nm : \text{Jac}(S) \rightarrow \text{Jac}(\Sigma)$ denote the norm map associated to $\pi : S \rightarrow \Sigma$. Then for any $L \in \text{Jac}(S)$ we have $Nm(L) = \det(\pi_* L) \otimes K^{n(n-1)/2}$ [3], hence we can alternatively characterise $\text{Prym}_{\bar{d}}(S, \Sigma)$ as those line bundles $L \in \text{Jac}_{\bar{d}}(S)$ with $Nm(L) = L_0 \otimes K^{n(n-1)/2}$.

3. HAMILTONIAN FLOWS OF THE HITCHIN SYSTEM

3.1. Flows on non-singular fibres. Given a holomorphic function $f : \mathcal{A} \rightarrow \mathbb{C}$ on \mathcal{A} , we let $X_f \in H^0(\mathcal{M}_{n,d}^{\text{sm}}, T\mathcal{M}_{n,d}^{\text{sm}})$ denote the corresponding Hamiltonian vector field on $\mathcal{M}_{n,d}^{\text{sm}}$, given by the relation:

$$i_{X_f} \Omega_I = h^*(df).$$

It will be useful to have a more explicit description of these Hamiltonian vector fields. Consider a point $b \in \mathcal{A}^{\text{reg}}$ and let $\pi : S \rightarrow \Sigma$ be the corresponding spectral curve. The fibre $h^{-1}(b)$ of the Hitchin system over b is $\text{Prym}_{\bar{d}}(S, \Sigma) \subset \text{Jac}_{\bar{d}}(S)$, so for each $m \in h^{-1}(b)$ the vertical tangent space $T_m h^{-1}(b)$ can be canonically identified with the kernel of $Nm_* : H^1(S, \mathcal{O}_S) \rightarrow H^1(\Sigma, \mathcal{O}_\Sigma)$. Under this identification $X_f(m)$ is an element of $H^1(S, \mathcal{O}_S)$.

Recall that $\mathcal{A} = \bigoplus_{j=2}^n H^0(\Sigma, K^j)$ and define $\mathcal{A}^* = \bigoplus_{j=2}^n H^1(\Sigma, K^{1-j})$. Serre duality applied termwise defines a dual pairing $\langle \cdot, \cdot \rangle : \mathcal{A}^* \otimes \mathcal{A} \rightarrow \mathbb{C}$.

Proposition 3.1. *Given $b = (b_2, \dots, b_n) \in \mathcal{A}^{\text{reg}}$, let $\gamma_b : \mathcal{A}^* = \bigoplus_{j=2}^n H^1(\Sigma, K^{1-j}) \rightarrow H^1(S, \mathcal{O}_S)$ be given by*

$$\gamma_b \left(\sum_{j=2}^n \mu_j \right) = \sum_{j=2}^n \pi^*(\mu_j) \left(\lambda^{j-1} - \frac{(j-1)\pi^*(b_{j-1})}{n} \right),$$

Where $\mu_j \in H^1(\Sigma, K^{1-j})$. Then for any $f \in \mathcal{O}(\mathcal{A})$ and any $m \in h^{-1}(b)$, we have

$$(3.1) \quad X_f(m) = \gamma_b(df(b)).$$

Proof. Let (E, Φ) be a Higgs bundle representing the point $m \in h^{-1}(b)$. Let $(\dot{A}, \dot{\Phi}) \in T_m \mathcal{M}_{n,d}^{\text{reg}}$ be a tangent vector at m . Differentiating the Hitchin map gives $h_*(\dot{A}, \dot{\Phi}) = (Tr(\Phi \dot{\Phi}), \dots, Tr(\Phi^{n-1} \dot{\Phi}))$. Let f be a holomorphic function on \mathcal{A} and set $df(m) = \sum_{j=2}^n [\mu_j]$ where $\mu_j \in \Omega^{0,1}(\Sigma, K^{1-j})$ represents a class

$[\mu_j] \in H^1(\Sigma, K^{1-j})$. We find

$$\begin{aligned}
\Omega_I(X_f(m), (\dot{A}, \dot{\Phi})) &= \langle df(m), h_*(\dot{A}, \dot{\Phi}) \rangle \\
&= \left\langle \sum_{j=2}^n \mu_j, h_*(\dot{A}, \dot{\Phi}) \right\rangle \\
&= \sum_{j=2}^n \int_{\Sigma} \mu_j Tr(\Phi^{j-1} \dot{\Phi}) \\
&= \sum_{j=2}^n \int_{\Sigma} Tr(\Phi^{j-1} \mu_j \dot{\Phi}) \\
&= \Omega_I \left(\left(\sum_{j=2}^n (\Phi^{j-1})_0 \mu_j, 0 \right), (\dot{A}, \dot{\Phi}) \right)
\end{aligned}$$

where $(\Phi^{j-1})_0$ denotes the trace-free part of Φ^{j-1} . Note that $Tr(\Phi^{j-1}) = (j-1)b_{j-1}$, so that $(\Phi^{j-1})_0 = \Phi^{j-1} - \frac{(j-1)b_{j-1}}{n} Id$. It follows that $X_f(m)$ is represented by a pair $(\dot{A}, \dot{\Phi})$ of the form $(\dot{A}, \dot{\Phi}) = (\sum_{j=2}^n (\Phi^{j-1} - \frac{(j-1)b_{j-1}}{n} Id) \mu_j, 0)$. Now suppose that L is the line bundle on S corresponding to (E, Φ) , so $E = \pi_*(L)$ and Φ is obtained by pushing forward $\lambda : L \rightarrow L \otimes \pi^*(K)$. Observe that $\dot{A} = \sum_{j=2}^n (\Phi^{j-1} - \frac{(j-1)b_{j-1}}{n} Id) \mu_j$ is obtained by pushing forward $\sum_{j=2}^n \pi^*(\mu_j)(\lambda^{j-1} - \frac{(j-1)b_{j-1}}{n})$, viewed as a deformation of the holomorphic structure on L . Equation (3.1) follows immediately. \square

Remark 3.2. If f is any global holomorphic on \mathcal{A} , then the corresponding Hamiltonian vector field X_f is defined on all of $\mathcal{M}_{n,d}^{\text{sm}}$. For any point $m = (\bar{\partial}_E, \Phi) \in \mathcal{M}_{n,d}^{\text{sm}}$ we have

$$X_f(m) = (\dot{A}, \dot{\Phi}) = \left(\sum_{j=2}^n \mu_j \left(\Phi^{j-1} - \frac{(j-1)b_{j-1}}{n} Id \right), 0 \right),$$

where $df(b) = \sum_{j=2}^n \mu_j$ and $b = \pi(m)$. Moreover we can integrate X_f to a biholomorphism $e^{X_f} : \mathcal{M}_{n,d}^{\text{sm}} \rightarrow \mathcal{M}_{n,d}^{\text{sm}}$ given by

$$e^{X_f}(\bar{\partial}_E, \Phi) = \left(\bar{\partial}_E + \sum_{j=2}^n \mu_j \left(\Phi^{j-1} - \frac{(j-1)b_{j-1}}{n} Id \right), \Phi \right).$$

The main point to note is that e^{X_f} so defined, preserves semi-stability and preserves the S -equivalence relation. This is clear because (E, Φ) and $e^{X_f}(E, \Phi)$ have the same Φ -invariant sub-bundles. It is also clear that e^{X_f} even extends to an automorphism on the whole of $\mathcal{M}_{n,d}$.

Lemma 3.3. *Given $b = (b_2, \dots, b_n) \in \mathcal{A}^{\text{reg}}$, let h_i denote the complete homogeneous symmetric polynomials, defined in Equation (2.4). Then for any $\tau \in \Omega^{0,1}(\Sigma, K^{n-r})$ with $r \geq 0$ we have*

$$\int_S \pi^*(\tau) \lambda^r = \begin{cases} 0 & \text{if } r < n-1, \\ \int_{\Sigma} \tau & \text{if } r = n-1, \\ \int_{\Sigma} \tau h_{r-n+1} & \text{if } r \geq n. \end{cases}$$

Proof. Using a partition of unity and the fact that the fibres of π containing branch points have measure zero, it suffices to consider the case that τ is supported in an open set $U \subseteq \Sigma$ which contains no branch points and such that $\pi^{-1}(U) \rightarrow U$ is the trivial n -fold covering. The derivative $d\pi$ of π is a holomorphic section of $K_S \pi^*(K^{-1}) = \pi^*(K^{n-1})$. If the characteristic equation for S is given by $\lambda^n + a_2 \lambda^{n-2} + \dots + a_n = 0$ then we claim that

$$(3.2) \quad d\pi = n\lambda^{n-1} + (n-2)a_2\lambda^{n-3} + \dots + a_{n-1}.$$

Indeed both sides of (3.2) have the same divisor, so there is a unique isomorphism $K_S \cong \pi^*(K^n)$ for which (3.2) holds. Since the covering $\pi^{-1}(U) \rightarrow U$ is trivial, the characteristic polynomial may be factored as $(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$, where the roots λ_i are holomorphic sections of $K|_U$. Then

$$d\pi = \sum_{j=1}^n (\lambda - \lambda_1) \cdots (\widehat{\lambda - \lambda_j}) \cdots (\lambda - \lambda_n),$$

where we use $\widehat{\lambda - \lambda_i}$ to denote omission of the factor $\lambda - \lambda_i$. It follows that

$$\begin{aligned} \int_{\pi^{-1}(U)} \pi^*(\tau) \lambda^r &= \int_U \tau \sum_{i=1}^n \frac{\lambda_i^r}{d\pi|_{\lambda=\lambda_i}} \\ &= \int_U \tau \sum_{i=1}^n \frac{\lambda_i^r}{\prod_{a \neq i} (\lambda_i - \lambda_a)}. \end{aligned}$$

For any given $u \in U$, we perform a contour integral in the fibre $K_u \cong \mathbb{C}$ over a contour containing all zeros $\lambda_1(u), \lambda_2(u), \dots, \lambda_n(u)$ of the characteristic polynomial at u . For $0 \leq r \leq n-1$ we find:

$$\sum_{i=1}^n \frac{\lambda_i^r}{\prod_{a \neq i} (\lambda_i - \lambda_a)} = \begin{cases} 0 & \text{if } r < n-1, \\ 1 & \text{if } r = n-1. \end{cases}$$

This proves the lemma for $0 \leq r \leq n-1$. The case $r \geq n$ can be proved inductively using the characteristic equation in the form $\lambda^n = -a_2 \lambda^{n-2} - a_3 \lambda^{n-3} - \dots - a_n$. For example when $r = n$, we have

$$\int_S \pi^*(\tau) \lambda^n = \int_S \pi^*(\tau) (-a_2 \lambda^{n-2} - a_3 \lambda^{n-3} - \dots - a_n) = 0.$$

In general for $i \geq 0$, we obtain an identity of the form $\int_S \pi^*(\tau) \lambda^{n-1+i} = \int_\Sigma \tau k_i$, where the k_i are given inductively by $k_0 = 1$, $k_i = -\sum_{j=1}^i a_j k_{i-j}$, $i \geq 1$. Comparing with (2.5), it follows that the $k_i = h_i$ are the complete homogeneous symmetric polynomials and this proves the lemma. \square

Proposition 3.4. *Given $b \in \mathcal{A}^{\text{reg}}$, let $\psi_b : \mathcal{A} \rightarrow H^0(S, K_S)$ be the composition*

$$\mathcal{A} \xrightarrow{(\gamma_b^t)^{-1}} H^1(S, \mathcal{O}_S)^* \xrightarrow{\cong} H^0(S, K_S) \xrightarrow{\cong} H^0(S, \pi^* K^n)$$

where the second arrow is Serre duality. Then for $\nu = \sum_{j=2}^n \nu_j \in \bigoplus_{j=2}^n H^0(\Sigma, K^j) = \mathcal{A}$ we have:

$$\psi_b(\nu) = \sum_{j=2}^n \pi^*(\nu_j) (\lambda^{n-j} + a_2 \lambda^{n-j-2} + \dots + a_{n-j}).$$

Proof. Let $\mu = \sum_{j=2}^n \mu_j \in \bigoplus_{j=2}^n H^1(\Sigma, K^{1-j}) = \mathcal{A}^*$. We need to show for all such μ that:

$$(3.3) \quad \int_S \gamma_b(\mu) \psi_b(\nu) = \langle \mu, \nu \rangle = \sum_{j=2}^n \int_\Sigma \mu_j \nu_j.$$

For this it is convenient to first introduce a map $\theta_b : \mathcal{A} \rightarrow H^0(S, K_S) \cong H^0(S, \pi^* K^n)$ which for $\nu_j \in H^0(\Sigma, K^j)$ is given by $\theta_b(\nu_j) = \pi^*(\nu_j) \lambda^{n-j}$. Using Lemma 3.3, we have:

$$\int_S \gamma_b(\mu_m) \theta_b(\nu_j) = \begin{cases} 0 & \text{if } m < j, \\ \int_\Sigma \mu_j \nu_j & \text{if } m = j, \\ \int_\Sigma \mu_m \nu_j h_{m-j}, & \text{if } m > j. \end{cases}$$

From this it follows easily that if we let $\psi_b(\nu_j) = \pi^*(\nu_j)(\lambda^{n-j} + a_2 \lambda^{n-j-2} + \dots + a_{n-j})$ then (3.3) is satisfied. \square

3.2. Flows on generic singular fibres. We will need to examine the Hamiltonian vector fields along the fibres of h lying over generic points of the discriminant divisor \mathcal{D} . As noted in [19], a generic point $b \in \mathcal{D}$ has a spectral curve $S \rightarrow \Sigma$ which has exactly one singular point, which is an ordinary double point. The argument used in [19] is as follows: let $a_n \in H^0(\Sigma, K^n)$ have a unique double zero $u \in \Sigma$. Then the spectral curve with characteristic equation $\lambda^n + a_n = 0$ has an ordinary double point lying over u and is smooth at all other points. Then by a semicontinuity argument there is a non-empty Zariski open subset $\mathcal{D}^0 \subset \mathcal{D}$ given by

$$\mathcal{D}^0 = \{b \in \mathcal{D} \mid S_b \text{ is irreducible and has a unique ordinary double point}\}.$$

Moreover, the discriminant divisor $\mathcal{D} \subset \mathcal{A}$ is irreducible ([19, Corollary 1.5]), so the complement $\mathcal{D} - \mathcal{D}^0$ has positive codimension in \mathcal{D} .

Consider now the spectral curve $\pi : S \rightarrow \Sigma$ associated to a point $b \in \mathcal{D}^0$. Let $p \in S$ be the singular point of S and $u = \pi(p)$. We let $\nu : S^\nu \rightarrow S$ be the normalization of S and $\nu^{-1}(p) = \{p_+, p_-\}$. Set $\pi^\nu = \pi \circ \nu$. According to [3], the fibre $h^{-1}(b)$ of $\mathcal{M}_{n,d}$ lying over b is a generalised Prym variety

$$h^{-1}(b) = \overline{\text{Prym}}_{d-n(g-1)}(S, \Sigma) = \{M \in \overline{\text{Jac}}_{d-n(g-1)}(S) \mid \det(\pi_* M) = L_0\},$$

where $\overline{\text{Jac}}_{d-n(g-1)}(S)$ is the generalised Jacobian consisting of rank 1, torsion-free sheaves on S with Euler characteristic $d - n(g-1)$. For our purposes it will enough to focus on the dense open subset $\overline{\text{Prym}}_{d-n(g-1)}^{\text{loc free}}(S, \Sigma) \subset \overline{\text{Prym}}_{d-n(g-1)}(S, \Sigma)$ of the fibre consisting of those $M \in \overline{\text{Prym}}_{d-n(g-1)}(S, \Sigma)$ which are locally free. Such M can be equivalently described as follows: start with a holomorphic line bundle L on S^ν and an isomorphism $\phi : L_{p_+} \rightarrow L_{p_-}$. The sheaf of holomorphic sections s of L for which $s(p_-) = \phi(s(p_+))$ defines a locally-free rank 1 sheaf M on S . Under this correspondence $M \in \overline{\text{Prym}}_{d-n(g-1)}(S, \Sigma)$ if and only if $\det(\pi^\nu)_* L = L_0(-u)$. If we fix the underlying \mathcal{C}^∞ line bundle L then a point in $\overline{\text{Prym}}_{d-n(g-1)}^{\text{loc free}}(S, \Sigma)$ can be represented by a pair $(\bar{\partial}_L, \phi)$, where $\bar{\partial}_L$ is a $\bar{\partial}$ -operator on L and ϕ is an isomorphism $\phi : L_{p_+} \rightarrow L_{p_-}$. A gauge transformation $g : S^\nu \rightarrow \mathbb{C}^*$ acts on such pairs as $g(\bar{\partial}_L, \phi) = (\bar{\partial}_L + g^{-1}dg, g(p_+)g(p_-)^{-1}\phi)$. Then $\overline{\text{Prym}}_{d-n(g-1)}^{\text{loc free}}(S, \Sigma)$ is identified with the set of equivalence classes of pairs $(\bar{\partial}_L, \phi)$, subject to the condition that $\det(\pi^\nu)_*(L, \bar{\partial}_L) = L_0(-u)$.

Lemma 3.5. *The canonical bundle K_{S^ν} of S^ν is isomorphic to $(\pi^\nu)^* K^n(-p_+ - p_-)$.*

Proof. Let $\rho : K_p \rightarrow K$ be the blow-up of the total space of K at the point p and $E = \rho^{-1}(p)$ the exceptional divisor. Then $S^\nu \subset K_p$ is the proper transform of S . Thus as divisors on K_p we have $S^\nu = \rho^* S - 2E$. Note that K is the cotangent bundle of Σ , so it has trivial canonical bundle. It follows that the canonical bundle of K_p is $[E]$ and by adjunction we have $K_{S^\nu} = ([S^\nu] + [E])|_{S^\nu} = (\rho^*[S] - [E])|_{S^\nu} = (\pi^\nu)^* K^n - [p_+ + p_-]$. That is, $K_{S^\nu} \cong (\pi^\nu)^* K^n(-p_+ - p_-)$. \square

Proposition 3.6. *For any $b \in \mathcal{D}^0$, the differential h_* of the Hitchin map is surjective at all points of $\overline{\text{Prym}}_{d-n(g-1)}^{\text{loc free}}(S, \Sigma)$.*

Proof. First note that $\overline{\text{Prym}}_{d-n(g-1)}(S, \Sigma) \subset \mathcal{M}_{n,d}^{\text{sm}}$. Indeed it can be shown that every point in $\overline{\text{Prym}}_{d-n(g-1)}(S, \Sigma)$ gives rise to a stable Higgs bundle [19, Remark 1.5]. Any $f \in \mathcal{A}^*$ may be thought of as a linear function on \mathcal{A} , hence has an associated Hamiltonian vector field X_f . To show that h_* is surjective on $\overline{\text{Prym}}_{d-n(g-1)}^{\text{loc free}}(S, \Sigma)$ it is clearly sufficient to show that for any non-zero $f \in \mathcal{A}^*$, X_f is non-vanishing on $\overline{\text{Prym}}_{d-n(g-1)}^{\text{loc free}}(S, \Sigma)$. To show this, we need to examine how the Hamiltonian flow of X_f acts on pairs $(\bar{\partial}_L, \phi)$. Given such a pair $(\bar{\partial}_L, \phi)$, let $F = (\pi^\nu)_* L$ and let $\Phi' : F \rightarrow F \otimes K$ be the endomorphism obtained by pushing forward $\nu^*(\lambda)$. The Higgs bundle (E, Φ) associated to $(\bar{\partial}_L, \phi)$ is related to (F, Φ') through a Hecke modification as follows: the isomorphism $\phi : L_{p_+} \rightarrow L_{p_-}$ defines a codimension 1 subspace $F_\phi \subset F_u$ of the fibre of F at u . Then $\mathcal{O}(E)$ is defined as the subsheaf of $\mathcal{O}(F)$ consisting of sections s for which $s(u) \in F_\phi$ (whenever u is in the domain of s). By construction Φ' preserves the subspace V_ϕ and the Higgs field Φ is simply the restriction of Φ' to $\mathcal{O}(E)$.

Writing $df = \sum_{j=2}^n f_j$, for $f_j \in \Omega^{0,1}(\Sigma, K^{1-j})$, we have by Remark 3.2 that the flow of X_f on $(\bar{\partial}_E, \Phi)$ is given by

$$e^{tX_f}(\bar{\partial}_E, \Phi) = \left(\bar{\partial}_E + t \sum_{j=2}^n f_j \left(\Phi^{j-1} - \frac{(j-1)b_{j-1}}{n} \text{Id} \right), \Phi \right).$$

At the level of pairs $(\bar{\partial}_L, \phi)$ it follows that the flow is given by

$$(\bar{\partial}_L, \phi) \mapsto \left(\bar{\partial}_L + t \sum_{j=2}^n (\pi^\nu)^*(f_j) \left(\nu^*(\lambda^{j-1}) - \frac{(j-1)(\pi^\nu)^*(b_{j-1})}{n} \right), \phi \right).$$

Now suppose that $f \in \mathcal{A}^*$ is non-zero and that X_f vanishes at $(\bar{\partial}_L, \phi)$. This is equivalent to requiring that

$$(3.4) \quad \sum_{j=2}^n (\pi^\nu)^*(f_j) \left(\nu^*(\lambda^{j-1}) - \frac{(j-1)(\pi^\nu)^*(b_{j-1})}{n} \right) = \bar{\partial}g,$$

for some $g : S^\nu \rightarrow \mathbb{C}$ satisfying $g(p_+) = g(p_-)$. For any $\mu_j \in \Omega^{0,1}(\Sigma, K^{1-j})$ let us define

$$\tilde{\gamma}_b \left(\sum_{j=2}^n \mu_j \right) = \sum_{j=2}^n (\pi^\nu)^*(\mu_j) \left(\nu^*(\lambda^{j-1}) - \frac{(j-1)(\pi^\nu)^*b_{j-1}}{n} \right) \in \Omega^{0,1}(S^\nu).$$

Furthermore, let us define $\tilde{\psi}_b : \mathcal{A} \rightarrow H^0(S^\nu, (\pi^\nu)^*(K^n))$ by

$$\tilde{\psi}_b(\nu_j) = (\pi^\nu)^*(\nu_j)(\nu^* \lambda^{n-j} + a_2 \nu^* \lambda^{n-j-2} + \dots + a_{n-j}).$$

By Lemma 3.5, $(\pi^\nu)^* K^n \cong K_{S^\nu}(p_+ + p_-)$. Thus we may interpret $H^0(S^\nu, (\pi^\nu)^*(K^n))$ as the space of meromorphic sections of K_{S^ν} which have at worst first order poles at p_+, p_- . For any $\mu_j \in \Omega^{0,1}(\Sigma, K^{1-j})$ and $\nu_m \in H^0(\Sigma, K^m)$ we have that $\tilde{\gamma}_b(\mu_j)\tilde{\psi}_b(\nu_m)$ can be viewed as a $(1,1)$ -form on S^ν away from the poles of $\tilde{\psi}_b(\nu_m)$. Now as the poles of $\tilde{\psi}_b(\nu_m)$ are first order it is easy to see that $\tilde{\gamma}_b(\mu_j)\tilde{\psi}_b(\nu_m)$ is integrable on S^ν . Moreover, since S^ν coincides with S away from a set of measure zero we find that

$$\int_{S^\nu} \tilde{\gamma}_b(\mu_j)\tilde{\psi}_b(\nu_m) = \int_S \gamma_b(\mu_j)\psi_b(\nu_m) = \langle \mu_j, \nu_m \rangle,$$

where as usual, $\langle \cdot, \cdot \rangle$ is the pairing of \mathcal{A}^* and \mathcal{A} . From Equation (3.4) we have that $\tilde{\gamma}_b(df) = \bar{\partial}g$, where g is a function on S^ν such that $g(p_+) = g(p_-)$. Thus, for any $\nu_m \in H^0(\Sigma, K^m)$ we have that:

$$\langle df, \nu_m \rangle = \int_{S^\nu} (\bar{\partial}g)\tilde{\psi}_b(\nu_m).$$

If $\tilde{\psi}_b(\nu_m)$ has no poles then this expression vanishes. More generally, suppose that $\tilde{\psi}_b(\nu_m)$ has first order poles at p_+, p_- with residues r_+, r_- . We have $r_+ + r_- = 0$, since the residues of a meromorphic 1-form on S^ν must sum to zero. Choose local coordinates z_+, z_- centered at p_+, p_- and let $D_+(\epsilon), D_-(\epsilon)$ be the corresponding discs of radius ϵ around p_+, p_- . Then

$$\begin{aligned} \int_{S^\nu} (\bar{\partial}g)\tilde{\psi}_b(\nu_m) &= \lim_{\epsilon \rightarrow 0} \int_{S^\nu - D_+(\epsilon) - D_-(\epsilon)} (\bar{\partial}g)\tilde{\psi}_b(\nu_m) \\ &= \lim_{\epsilon \rightarrow 0} \int_{S^\nu - D_+(\epsilon) - D_-(\epsilon)} d\left(g\tilde{\psi}_b(\nu_m)\right) \\ &= \lim_{\epsilon \rightarrow 0} \left(\int_{\partial D_+(\epsilon)} g\tilde{\psi}_b(\nu_m) + \int_{\partial D_-(\epsilon)} g\tilde{\psi}_b(\nu_m) \right) \\ &= 2\pi i(g(p_+)r_+ + g(p_-)r_-). \end{aligned}$$

But $g(p_+) = g(p_-)$, so this is $2\pi i g(p_+)(r_+ + r_-) = 0$. We have shown that $\langle df, \nu_m \rangle = 0$ for all ν_m , hence $df = 0$. But f is a linear function so this means $f = 0$. But we assumed that f is non-zero, hence X_f must be non-vanishing. \square

Corollary 3.7. *The set of points of $\mathcal{M}_{n,d}^{\text{sm}}$ where the differential of the Hitchin map is not surjective has codimension ≥ 2 .*

4. HOLOMORPHIC VECTOR FIELDS ON $\mathcal{M}_{n,d}^{\text{sm}}$

4.1. Kodaira-Spencer maps.

Lemma 4.1. *Let $S \rightarrow \Sigma$ be a non-singular spectral curve, where Σ has genus $g > 1$. Then S is not hyperelliptic.*

Proof. Suppose on the contrary, that S is hyperelliptic. Let $\sigma : S \rightarrow S$ be the hyperelliptic involution. Then σ acts on $H^0(S, K_S) \cong H^0(\Sigma, \pi^* K^n)$ as multiplication by -1 . As usual, we let λ denote the tautological section of $\pi^* K$. Then $s = \sigma^*(\lambda)$ is a section of $\sigma^*(\pi^* K)$. It follows that $s^n = \sigma^*(\lambda^n) = -\lambda^n$. Thus s^n and λ^n have

the same divisor. Of course this means s and λ also have the same divisor and $\sigma^*\pi^*K \cong \pi^*K$. It follows that we can lift σ to an isomorphism $\hat{\sigma} : \pi^*K \rightarrow \pi^*K$ covering σ and for which $\hat{\sigma}^{\otimes n} = \sigma^*$ is the pullback by σ on $\pi^*K^n \cong K_S$. But as σ is an involution, we have $\sigma^* \circ \sigma^* = 1$ and hence $\hat{\sigma} \circ \hat{\sigma} = cId$ for some constant c satisfying $c^n = 1$.

From the identity $\pi_*\mathcal{O}_S = \mathcal{O}_\Sigma \oplus \mathcal{O}(K^{-1}) \oplus \dots \oplus \mathcal{O}(K^{-n+1})$ [3], we see that $H^0(S, \pi^*K) \cong H^0(\Sigma, \mathcal{O}) \oplus H^0(\Sigma, K) \cong \mathbb{C} \oplus H^0(\Sigma, K)$. Thus $\hat{\sigma}(\lambda) = a\lambda + \pi^*(b)$, where $a \in \mathbb{C}$ and $b \in H^0(\Sigma, K)$. But now we have $-\lambda^n = \sigma^*(\lambda^n) = (a\lambda + b)^n$ and so $\lambda^n + (a\lambda + b)^n = 0$. Note that $\lambda^n + (a\lambda + b)^n$ factors as a polynomial in λ . This would contradict irreducibility of S unless the coefficients of this polynomial all vanish, so $a^n = -1$ and $b = 0$. Therefore $\hat{\sigma}(\lambda) = a\lambda$ and since $\hat{\sigma} \circ \hat{\sigma} = cId$, we need $a^2 = c$. Note in particular that $a^{2n} = 1$.

Now let $\alpha \in H^0(\Sigma, K^j)$. Then $\lambda^{(2n-1)j}\pi^*\alpha \in H^0(S, \pi^*K^{2nj}) \cong H^0(S, K_S^{2j})$. Therefore $\sigma^*(\lambda^{(2n-1)j}\pi^*\alpha) = \lambda^{(2n-1)j}\pi^*\alpha$ and so $a^{(2n-1)j}\hat{\sigma}^{\otimes j}(\pi^*\alpha) = \alpha$. Hence using $a^{2n} = 1$, we have $\hat{\sigma}^{\otimes j}(\pi^*\alpha) = a^j\pi^*\alpha$. Next, using $\mathcal{O}_S = \mathcal{O}_\Sigma \oplus \mathcal{O}(K^{-1}) \oplus \dots \oplus \mathcal{O}(K^{-n+1})$, we have $H^0(S, K_S^2) \cong H^0(\Sigma, K^{2n}) \oplus \dots \oplus H^0(\Sigma, K^{n+1})$ and any $\omega \in H^0(S, K_S^2)$ can be written as $\omega = \pi^*\omega_{2n} + \dots + \pi^*\omega_{n+1}\lambda^{n-1}$, where $\omega_j \in H^0(\Sigma, K^j)$. It follows that $\sigma^*(\omega) = a^{2n}\omega = \omega$, so that σ acts as the identity on $H^0(S, K_S^2)$. However this never happens for a hyperelliptic curve of genus > 2 . On the other hand the genus of S satisfies $g_S = 1 + n^2(g-1) > 2$, so this is a contradiction. \square

Recall that $\mathcal{M}_{n,d}$ admits an action of \mathbb{C}^* as follows: for any $\lambda \in \mathbb{C}^*$ we let $m_\lambda : \mathcal{M}_{n,d} \rightarrow \mathcal{M}_{n,d}$ be defined as $m_\lambda(E, \Phi) = (E, \lambda\Phi)$. There is a unique \mathbb{C}^* -action m_λ^A on \mathcal{A} compatible with the \mathbb{C}^* -action on $\mathcal{M}_{n,d}$ in the sense that $h \circ m_\lambda = m_\lambda^A \circ h$. Note that this \mathbb{C}^* -action preserves \mathcal{A}^{reg} . Under the decomposition $\mathcal{A} = \bigoplus_{j=2}^n H^0(\Sigma, K^j)$ we have that the subspace $H^0(\Sigma, K^j)$ has weight j with respect to this action. Let $\xi = \frac{d}{dt}|_{t=0} m_{e^t}$ ($t \in \mathbb{R}$) be the holomorphic vector field on $\mathcal{M}_{n,d}^{\text{sm}}$ tangent to the action of $\mathbb{R}_+ \subset \mathbb{C}^*$ and similarly let $\xi^A = \frac{d}{dt}|_{t=0} m_{e^t}^A$. It follows that $h_*\xi_m = \xi_{h(m)}^A$ for all $m \in \mathcal{M}_{n,d}^{\text{sm}}$.

Recall that to any point $b \in \mathcal{A}$ we associate a section s_b of $\pi^*(K^n)$ on the total space of K and that the corresponding spectral curve S_b is the zero locus of s_b . We can similarly construct the universal family $S_{\text{univ}}^{\text{reg}}$ of regular spectral curves:

$$S_{\text{univ}}^{\text{reg}} = \{(b, u) \in \mathcal{A}^{\text{reg}} \times K \mid s_b(u) = 0\}.$$

Let $q : S_{\text{univ}}^{\text{reg}} \rightarrow \mathcal{A}^{\text{reg}}$ be the projection $(b, u) \mapsto b$. It is clear that $S_{\text{univ}}^{\text{reg}}$ is smooth and that the fibre of q over b is precisely the spectral curve S_b . Thus for any $b \in \mathcal{A}^{\text{reg}}$ we have a Kodaira-Spencer map $\rho_b : \mathcal{A} \cong T_b\mathcal{A}^{\text{reg}} \rightarrow H^1(S_b, TS_b)$.

Lemma 4.2. *For any $b \in \mathcal{A}^{\text{reg}}$, the kernel of the Kodaira-Spencer map $\rho_b : \mathcal{A} \rightarrow H^1(S_b, TS_b)$ is spanned by ξ_b^A .*

Proof. Denote S_b more simply as S . On S we have an exact sequence

$$(4.1) \quad 0 \rightarrow TS \rightarrow TK|_S \xrightarrow{\beta} N \rightarrow 0,$$

where N is the normal bundle to S in K . The Kodaira-Spencer map fits into the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H^0(S, TK|_S) & \xrightarrow{\beta} & H^0(S, N) & \xrightarrow{\delta} & H^1(S, TS) \\ & & & & \uparrow \chi & \nearrow \rho_b & \\ & & & & \mathcal{A} & & \end{array}$$

where the horizontal sequence of maps is obtained from the long exact sequence associated to (4.1) and $\chi : \mathcal{A} \rightarrow H^0(S, N)$ is the characteristic map [12]. We have $N = [S]|_S = \pi^* K^n$, hence $H^0(S, N) = H^0(S, \pi^* K^n) = \bigoplus_{j=1}^n H^0(\Sigma, K^j)$. One sees that the map χ is the obvious inclusion $\mathcal{A} = \bigoplus_{j=2}^n H^0(\Sigma, K^j) \subset \bigoplus_{j=1}^n H^0(\Sigma, K^j)$. By exactness of the horizontal sequence we have that $\text{Ker}(\rho_b) = \text{Ker}(\delta) \cap \text{Im}(\chi) = \text{Im}(\beta) \cap \text{Im}(\chi)$.

Observe that on the total space of K there is an exact sequence $0 \rightarrow \pi^* K \rightarrow TK \rightarrow \pi^* K^{-1} \rightarrow 0$. Restricting to S and taking the associated long exact sequence gives:

$$0 \rightarrow H^0(S, \pi^* K) \rightarrow H^0(S, TK|_S) \rightarrow H^0(S, \pi^* K^{-1}).$$

Now using $\pi_* \mathcal{O}_S = \mathcal{O}_\Sigma \oplus \mathcal{O}(K^{-1}) \oplus \dots \oplus \mathcal{O}(K^{-n+1})$, we see that $H^0(S, \pi^* K^{-1}) = 0$. Therefore we have an isomorphism $H^0(S, TK|_S) \cong H^0(S, \pi^* K)$. Moreover, we have $H^0(S, \pi^* K) \cong H^0(\Sigma, \mathcal{O}) \oplus H^0(\Sigma, K)$. In fact, it is easy to see what the corresponding sections on $H^0(S, TK|_S)$ are. The factor $H^0(\Sigma, \mathcal{O}) \cong \mathbb{C}$ is spanned by the vector field generating the \mathbb{C}^* -action in the fibres of $K \rightarrow \Sigma$. An element $\alpha \in H^0(\Sigma, K)$ defines a section of $TK|_S$ whose value at $s \in S$ is $\alpha(\pi(s)) \in (\pi^* K)_s \subset (TK)_s$. Given an element $(c, \alpha) \in H^0(\Sigma, \mathcal{O}) \oplus H^0(\Sigma, K)$ it is easy to see that $\beta(c, \alpha)$ is in the image of χ only if $\alpha = 0$ (because our spectral curves have no a_1 coefficient in their characteristic polynomial). This leaves a 1-dimensional space of deformations of S in K generated by the \mathbb{C}^* -action in the fibres of K . As this corresponds to the \mathbb{C}^* -action on \mathcal{A} , we have shown that the kernel of ρ_b is indeed spanned by $\xi_b^{\mathcal{A}}$. \square

In a similar fashion we can view $h : \mathcal{M}_{n,d}^{\text{reg}} \rightarrow \mathcal{A}^{\text{reg}}$ as a family of abelian varieties, hence to any $b \in \mathcal{A}^{\text{reg}}$ we have a Kodaira-Spencer map $\theta_b : \mathcal{A} \rightarrow H^1(h^{-1}(b), T h^{-1}(b))$.

Lemma 4.3. *Let Y be a holomorphic vector field on \mathcal{A}^{reg} such that $\theta_b(Y_b) = 0$ for all $b \in \mathcal{A}^{\text{reg}}$. Then $Y = f \xi^{\mathcal{A}}$ for some holomorphic function f on \mathcal{A}^{reg} .*

Proof. Let $\text{Jac}_{\tilde{d}}(S)$ be the degree \tilde{d} component of the Picard variety of S and $h^{-1}(b) = \text{Prym}_{\tilde{d}}(S, \Sigma)$ the Prym variety. We can in the same way define a Kodaira-Spencer map $\tau_b : \mathcal{A} \rightarrow H^1(\text{Jac}(S), T \text{Jac}(S))$. Let $Z = \text{Prym}_{\tilde{d}} \times \text{Jac}(\Sigma)$. The map $p : Z \rightarrow \text{Jac}_{\tilde{d}}(S)$ given by $p(M, N) = M \otimes \pi^*(N)$ is a covering space with fibre $\text{Jac}(\Sigma)[n]$, the points of order n in $\text{Jac}(\Sigma)$. Furthermore $TZ \cong p^*(T \text{Jac}_{\tilde{d}}(S)) \simeq \mathbb{C}^{2gs}$ and by Hodge theory the pullback $p^* : H^1(\text{Jac}_{\tilde{d}}(S), T \text{Jac}_{\tilde{d}}(S)) \rightarrow H^1(Z, TZ)$ is an isomorphism. It follows that if $\theta_b(Y_b) = 0$ then the corresponding deformation of $\text{Jac}_{\tilde{d}}(S)$ is trivial and so is the deformation of $\text{Jac}(S)$. This means that $\tau_b(Y_b) = 0$. Next observe that there is a natural map $J : H^1(S, TS) \rightarrow H^1(\text{Jac}(S), T \text{Jac}(S))$ such that $\tau_b = J \circ \rho_b$. By Lemma 4.1, S is not hyperelliptic and as is well known,

this implies that J is injective. Therefore $\tau_b(Y_b) = 0$ implies that $\rho_b(Y_b) = 0$ and by Lemma 4.2, Y is a multiple of ξ_b^A . \square

4.2. Classification of holomorphic vector fields. The restriction $h : \mathcal{M}_{n,d}^{\text{reg}} \rightarrow \mathcal{A}^{\text{reg}}$ of the Hitchin fibration over \mathcal{A}^{reg} is a smooth fibre bundle, so we have an exact sequence:

$$0 \longrightarrow \text{Ker}(h_*) \longrightarrow T\mathcal{M}_{n,d}^{\text{reg}} \xrightarrow{h_*} \mathcal{A} \longrightarrow 0,$$

where \mathcal{A} is thought of as a trivial vector bundle on $\mathcal{M}_{n,d}^{\text{reg}}$. The map sending a linear function $f \in \mathcal{A}^*$ to the corresponding Hamiltonian vector field X_f gives an isomorphism $\text{Ker}(h_*) \cong \mathcal{A}^*$, hence our exact sequence becomes:

$$(4.2) \quad 0 \longrightarrow \mathcal{A}^* \longrightarrow T\mathcal{M}_{n,d}^{\text{reg}} \xrightarrow{h_*} \mathcal{A} \longrightarrow 0.$$

Proposition 4.4. *Let X be a holomorphic vector field on $\mathcal{M}_{n,d}^{\text{reg}}$. There is a holomorphic function f on \mathcal{A}^{reg} such that for any $m \in \mathcal{M}_{n,d}^{\text{reg}}$ we have $h_*X_m = f(b)\xi_b^A$, where $b = h(m)$.*

Proof. From (4.2) we see that h_*X is a section of the trivial bundle \mathcal{A} and therefore must be constant over the fibres of h . So there is a holomorphic vector field Y on \mathcal{A}^{reg} such that $h_*X_m = Y_{h(m)}$ for all $m \in \mathcal{M}_{n,d}^{\text{reg}}$. We need to show that $Y = f\xi^A$ for some holomorphic function f on \mathcal{A}^{reg} .

Let $b \in \mathcal{A}^{\text{reg}}$ and let $\epsilon > 0$ be such that there exists an integral curve $\rho_b(t) : (-\epsilon, \epsilon) \rightarrow \mathcal{A}^{\text{reg}}$ of Y with $\gamma_b(0) = b$. The integral curves of X through points of $h^{-1}(b)$ must lie over $\gamma_b(t)$. This together with properness of the Hitchin map ensures that for every $m \in h^{-1}(b)$, the integral curve $\hat{\gamma}_m(t)$ of X with $\hat{\gamma}_m(0) = m$ exists for t in the interval $(-\epsilon, \epsilon)$. The fact that X is holomorphic now implies that the fibres $h^{-1}(\gamma_b(t))$ over $\gamma_b(t)$ are all biholomorphic to $h^{-1}(b)$. Therefore Y_b is in the Kernel of the Kodaira-Spencer map θ_b . Hence by Lemma 4.3, Y has the expected form. \square

We now give a construction of a large family of holomorphic vector fields on $\mathcal{M}_{n,d}^{\text{reg}}$. Let μ be a holomorphic 1-form on \mathcal{A}^{reg} . Then we define a holomorphic vector field X_μ on $\mathcal{M}_{n,d}^{\text{reg}}$ by the relation

$$i_{X_\mu}\Omega_I = h^*\mu.$$

Note that if μ extends to a holomorphic 1-form on all of \mathcal{A} , then X_μ extends to all of $\mathcal{M}_{n,d}^{\text{sm}}$.

Theorem 4.5. *Let X be a holomorphic vector field on $\mathcal{M}_{n,d}^{\text{reg}}$. Then there exists a holomorphic function f and holomorphic 1-form μ on \mathcal{A}^{reg} such that*

$$X = h^*(f)\xi + X_\mu.$$

Conversely, for every such f, μ we obtain a holomorphic vector field $X = h^(f)\xi + X_\mu$. Moreover X extends to $\mathcal{M}_{n,d}^{\text{sm}}$ if and only if f and μ extend to \mathcal{A} .*

Proof. Let X be a holomorphic vector field on $\mathcal{M}_{n,d}^{\text{reg}}$. By Proposition 4.4, there is a holomorphic function f on \mathcal{A}^{reg} such that $h_*(X_m) = f(b)\xi_b^A$, where $b = h(m)$. Consider the holomorphic vector field $Y = X - h^*(f)\xi$ on $\mathcal{M}_{n,d}^{\text{reg}}$. By construction $h_*(Y) = 0$, so Y is valued in $\text{Ker}(h_*)$, which is a trivial bundle isomorphic to \mathcal{A}^* .

It follows that Y is constant along the fibres of h , hence $Y = X_\mu$ for a unique holomorphic 1-form on \mathcal{A}^{reg} . This shows that $X = h^*(f)\xi + X_\mu$ as required. It is clear that if f and μ extend to \mathcal{A} , then X extends to $\mathcal{M}_{n,d}^{\text{sm}}$.

Conversely, suppose that X extends to $\mathcal{M}_{n,d}^{\text{sm}}$. It follows that f extends to \mathcal{A} , because $h : \mathcal{M}_{n,d}^{\text{sm}} \rightarrow \mathcal{A}$ is surjective, and $h_*(X_m) = f(b)\xi_b^{\mathcal{A}}$ for any $m \in \mathcal{M}_{n,d}^{\text{reg}}$ and $b = h(m)$. Let $U \subset \mathcal{M}_{n,d}^{\text{sm}}$ be the points where h_* is surjective and set $Y = X - h^*(f)\xi$. Since X and $h^*(f)$ extend to $\mathcal{M}_{n,d}^{\text{sm}}$, Y also extends to $\mathcal{M}_{n,d}^{\text{sm}}$ and satisfies $h_*(Y) = 0$. It follows that there is a holomorphic \mathcal{A}^* -valued function α on U such that $i_{Y_m}(\Omega_I)_m = h^*(\alpha(m))$ for any $m \in U$. In particular this means that α is an extension of $h^*(\mu)$ from $\mathcal{M}_{n,d}^{\text{reg}}$ to U . Now suppose that b belongs to \mathcal{D}^0 . By Proposition 3.6, there is a point $m \in h^{-1}(b)$ such that h_* is surjective at m . Note that since Y commutes with the Hamiltonian flows, we may find an open neighborhood $N \subseteq \mathcal{A}$ with $b \in N$ and a holomorphic \mathcal{A}^* -valued function μ' on N such that $\alpha = h^*(\mu')$ in a neighborhood of m . Clearly μ' must agree with μ on $N \cap (\mathcal{A} - \mathcal{D})$. Since this happens for all $b \in \mathcal{D}^0$, we have shown that μ extends to $\mathcal{A} - \mathcal{E}$, where $\mathcal{E} = \mathcal{D} - \mathcal{D}^0$. But \mathcal{E} has codimension at least 2 in \mathcal{A} , so in fact μ extends to the whole of \mathcal{A} . \square

Definition 4.6. By a 1-parameter subgroup ϕ_t of automorphisms of $\mathcal{M}_{n,d}$, we mean a group homomorphism $(\mathbb{R}, +) \rightarrow \text{Aut}(\mathcal{M}_{n,d}), t \mapsto \phi_t$. If X is a holomorphic vector field on $\mathcal{M}_{n,d}^{\text{sm}}$, we say that X integrates to ϕ_t if X integrates to the restriction of ϕ_t to $\mathcal{M}_{n,d}^{\text{sm}}$ in the usual sense.

Proposition 4.7. Let X be a holomorphic vector field on $\mathcal{M}_{n,d}^{\text{sm}}$, which by Theorem 4.5 can be written in the form $X = h^*(f)\xi + X_\mu$ with f a holomorphic function on \mathcal{A} and μ a holomorphic 1-form on \mathcal{A} . Then X integrates to a 1-parameter subgroup ϕ_t of $\text{Aut}(\mathcal{M}_{n,d})$ if and only if f is constant.

Proof. Suppose that $X = h^*(f)\xi + X_\mu$ integrates to a 1-parameter subgroup ϕ_t of automorphisms of $\mathcal{M}_{n,d}$. Since the only global holomorphic functions on $\mathcal{M}_{n,d}$ are those of the form $h^*(g)$, where g is a holomorphic function on \mathcal{A} , we see that there is a 1-parameter subgroup ψ_t of automorphisms of \mathcal{A} such that $h \circ \phi_t = \psi_t \circ h$. It follows that $f\xi^{\mathcal{A}}$ integrates to ψ_t .

Choose a point $b \in \mathcal{A}$ for which \mathbb{C}^* acts freely (or with kernel ± 1 in the case $n = 2$). Let $\mathcal{O}_b \cong \mathbb{C}^*$ be the orbit. Then ψ_t restricts to a 1-parameter family of automorphisms of \mathcal{O}_b . Any automorphism of \mathbb{C}^* is either of the form $z \mapsto cz$ or $z \mapsto cz^{-1}$, where $c \in \mathbb{C}^*$. In our case ψ_t is connected to the identity, so the automorphisms ψ_t of \mathcal{O}_b must be of the form $z \mapsto m_{e^{at}}z$ for some $a \in \mathbb{C}$. Thus for any $b \in \mathcal{A}$ for which the stabiliser of the \mathbb{C}^* -action is trivial (or ± 1 in the case $n = 2$) we have deduced that $\psi_t(b) = m_{e^{at}}b$, for some $a \in \mathbb{C}$. In fact it is clear that $a = f(b)$. What we have shown is that f is constant on the orbit \mathcal{O}_b . Since the closure of any such orbit contains the origin $0 \in \mathcal{A}$, we see that f must be a constant.

Conversely, suppose that $X = f\xi + X_\mu$, where f is constant. In the case that $f = 0$, it is easy to integrate X . In fact if $(\bar{\partial}_E, \Phi) \in \mathcal{M}_{n,d}$ and $b = h(\bar{\partial}_E, \Phi)$ then

$$e^{tX}(\bar{\partial}_E, \Phi) = \left(\bar{\partial}_E + t \sum_{j=2}^n \mu_j(b)(\Phi^{j-1} - \frac{(j-1)b_{j-1}}{n} Id), \Phi \right),$$

where $\mu_j(b)$ is the component of $\mu(b)$ in $H^1(\Sigma, K^{1-j})$. More generally, when f is non-zero we find,

$$e^{tX}(\bar{\partial}_E, \Phi) = \left(\bar{\partial}_E + \sum_{j=2}^n \alpha_{j,t}(b) (\Phi^{j-1} - \frac{(j-1)b_{j-1}}{n} Id), e^{tf}\Phi \right),$$

where $\alpha_{j,t}(b)$ is given by

$$\alpha_{j,t}(b) = \int_0^t e^{(j-1)uf} \mu_j(e^{uf}b) du.$$

□

4.3. Further properties of the holomorphic vector fields.

Proposition 4.8. *Let μ, ν be holomorphic 1-forms on \mathcal{A} . We have:*

$$(4.3) \quad [X_\mu, X_\nu] = 0,$$

$$(4.4) \quad [\xi, X_\mu] = X_\tau,$$

where $\tau = \mathcal{L}_{\xi\mathcal{A}}(\mu) - \mu$.

Proof. Starting with the identity $\mathcal{L}_{X_\mu}(i_{X_\nu}\Omega_I) = i_{[X_\mu, X_\nu]}\Omega_I + i_{X_\nu}\mathcal{L}_{X_\mu}\Omega_I$, we have

$$\begin{aligned} i_{[X_\mu, X_\nu]}\Omega_I &= \mathcal{L}_{X_\mu}(i_{X_\nu}\Omega_I) - i_{X_\nu}\mathcal{L}_{X_\mu}\Omega_I \\ &= \mathcal{L}_{X_\mu}(h^*\nu) - i_{X_\nu}(dh^*\mu) \\ &= i_{X_\mu}(h^*(d\nu)) - i_{X_\nu}(h^*(d\mu)) \\ &= 0, \end{aligned}$$

where we have used $h_*X_\mu = h_*X_\nu = 0$. This proves (4.3).

Observe that $m_\lambda^*\Omega_I = \lambda\Omega_I$, for any $\lambda \in \mathbb{C}^*$. Thus $\mathcal{L}_\xi\Omega_I = \Omega_I$. Consider now the following computation:

$$\begin{aligned} i_{[\xi, X_\mu]}\Omega_I &= \mathcal{L}_\xi(i_{X_\mu}\Omega_I) - i_{X_\mu}\mathcal{L}_\xi\Omega_I \\ &= \mathcal{L}_\xi(h^*(\mu)) - i_{X_\mu}\Omega_I \\ &= h^*(\mathcal{L}_{\xi\mathcal{A}}(\mu)) - h^*(\mu) \\ &= h^*(\mathcal{L}_{\xi\mathcal{A}}(\mu) - \mu). \end{aligned}$$

This proves (4.4), where $\tau = \mathcal{L}_{\xi\mathcal{A}}(\mu) - \mu$. □

Corollary 4.9. *Let μ be a holomorphic 1-form on \mathcal{A} . Then*

$$(e^{X_\mu})_*\xi = \xi + X_\tau,$$

where $\tau = \mathcal{L}_{\xi\mathcal{A}}(\mu) - \mu$.

Proof. Let $\xi_t = (e^{tX_\mu})_*\xi$. Then $\xi_0 = \xi$ and $\frac{d\xi_t}{dt} = [\xi, X_\mu] = X_\tau$, by Proposition 4.8. The result follows by integration. □

Corollary 4.10. *Let μ be a holomorphic 1-form on \mathcal{A} . The automorphism $e^{X_\mu} : \mathcal{M}_{n,d} \rightarrow \mathcal{M}_{n,d}$ commutes with the \mathbb{C}^* -action of and only if $\mu = 0$. In particular e^{X_μ} is the identity if and only if $\mu = 0$.*

Proof. As the \mathbb{C}^* -action is generated by ξ , we have that e^{X_μ} commutes with this action if and only if $(e^{X_\mu})_*\xi = \xi$. By Corollary 4.9, this happens if and only if $\mathcal{L}_{\xi^A}(\mu) = \mu$, that is, μ is invariant under the \mathbb{C}^* -action on \mathcal{A} . It is easy to see that the only invariant 1-form is $\mu = 0$. The second part of the Corollary follows, since the identity commutes with the \mathbb{C}^* -action. \square

Let $H^0(\mathcal{A}, \Omega^1(\mathcal{A}))$ be the space of holomorphic 1-forms on \mathcal{A} , which is an abelian group under addition. Consider the map $e^{X(\cdot)} : H^0(\mathcal{A}, \Omega^1(\mathcal{A})) \rightarrow \text{Aut}(\mathcal{M}_{n,d})$ sending μ to e^{X_μ} . Equation (4.3) implies that $e^{X_\mu}e^{X_\nu} = e^{X_\mu+X_\nu} = e^{X_{\mu+\nu}}$, so this map is a group homomorphism. Moreover, Corollary 4.10 implies that the map is injective. Thus we have identified $H^0(\mathcal{A}, \Omega^1(\mathcal{A}))$ as a subgroup of $\text{Aut}(\mathcal{M}_{n,d})$. We will denote the image of $H^0(\mathcal{A}, \Omega^1(\mathcal{A}))$ in $\text{Aut}(\mathcal{M}_{n,d})$ by $\text{Vert}_0(\mathcal{M}_{n,d})$ and call it the group of vertical translations of $\mathcal{M}_{n,d}$ connected to the identity. Indeed, on any non-singular fibre $h^{-1}(b)$ of $\mathcal{M}_{n,d}$, an element e^{X_μ} of this group acts as a translation in $h^{-1}(b)$.

5. PROOF OF THE MAIN THEOREM

Lemma 5.1. *For every holomorphic 1-form μ on \mathcal{A} there is a unique holomorphic 1-form ν on \mathcal{A} satisfying*

$$\mathcal{L}_{\xi^A}(\nu) - \nu = \mu.$$

Proof. Let z^1, \dots, z^r be linear coordinates on \mathcal{A} such that z^i has weight m_i with respect to the \mathbb{C}^* -action. This means that $\xi^A = m_1 z^1 \frac{\partial}{\partial z^1} + \dots + m_r z^r \frac{\partial}{\partial z^r}$. Moreover, since the subspace $H^0(\Sigma, K^j) \subset \mathcal{A}$ has weight j , we see that $m_i \geq 2$ for all i . Let ν be a holomorphic 1-form on \mathcal{A} . Then $\nu = \nu_1(z)dz^1 + \dots + \nu_r(z)dz^r$, where ν_1, \dots, ν_r are holomorphic functions on \mathcal{A} . Let $\mu = \mathcal{L}_{\xi^A}(\nu) - \nu$. Then $\mu = \mu_1(z)dz^1 + \dots + \mu_r(z)dz^r$, where

$$(5.1) \quad \mu_i(z) = \xi^A(\nu_i(z)) + (m_i - 1)\nu_i(z).$$

Given $\mu_i(z)$, we wish to find a solution $\nu_i(z)$ to (5.1). For each i , consider the function $\nu_i(z)$ defined by

$$\nu_i(z) = \left(\frac{1}{2\pi i}\right)^r \int_{|w^1|=\zeta^1} \dots \int_{|w^r|=\zeta^r} \mu_i(w) \left(\sum_I \frac{1}{(\sum_j m_j i_j + m_i - 1)} \left(\frac{z}{w}\right)^I \right) \frac{dw^1}{w^1} \dots \frac{dw^r}{w^r},$$

where the sum \sum_I is over multi-indices $I = (i_1, i_2, \dots, i_r)$ and for a given $z \in \mathcal{A}$, ζ_1, \dots, ζ_r are chosen large enough that $\sum_I \left(\frac{z}{w}\right)^I$ converges absolutely, e.g., $\zeta^i > |z^i|$ suffices. Note crucially that the denominator $(\sum_j m_j i_j + m_i - 1)$ is always ≥ 1 because $m_i \geq 2$ for all i . Clearly $\nu_i(z)$ is a globally defined holomorphic function. It is easy to check that $\nu_i(z)$ satisfies Equation (5.1). Thus $\nu = \nu_1(z)dz^1 + \dots + \nu_r(z)dz^r$ is a solution to $\mathcal{L}_{\xi^A}(\nu) - \nu = \mu$. Uniqueness follows for if ν is a holomorphic 1-form with $\mathcal{L}_{\xi^A}(\nu) - \nu = 0$, then ν is invariant under the \mathbb{C}^* -action and as in the proof of Corollary 4.10, this implies $\nu = 0$. \square

Proposition 5.2. *Let $\phi : \mathcal{M}_{n,d} \rightarrow \mathcal{M}_{n,d}$. There is a unique holomorphic 1-form ν on \mathcal{A} such that the composition $e^{X_\nu} \circ \phi$ commutes with the \mathbb{C}^* -action.*

Proof. Let $U \subset \mathcal{M}_{n,d}^{\text{sm}}$ be the points of $\mathcal{M}_{n,d}^{\text{sm}}$ where h_* is surjective. By Corollary 3.7, the complement of U in $\mathcal{M}_{n,d}^{\text{sm}}$ has codimension ≥ 2 . For any $m \in \mathcal{M}_{n,d}^{\text{sm}}$ let $A_m \subset T_m^* \mathcal{M}_{n,d}^{\text{sm}}$ be the subspace spanned by differentials $dg(m)$, where g is a

holomorphic function on $\mathcal{M}_{n,d}$. Since all holomorphic functions on $\mathcal{M}_{n,d}$ are pull-backs from \mathcal{A} it is easy to see that U is precisely the set of $m \in \mathcal{M}_{n,d}^{\text{sm}}$ such that $\dim(A_m) = \frac{1}{2}\dim(\mathcal{M}_{n,d})$. Now if g is any holomorphic function on $\mathcal{M}_{n,d}$ then ϕ^*g is also holomorphic. It follows that ϕ preserves U and sends A_m to $A_{\phi^{-1}(m)}$ under pullback of 1-forms. But for any $m \in U$, we have that $A_m \subset T_m^*\mathcal{M}_{n,d}^{\text{sm}}$ is the annihilator of $\text{Ker}(h_*) \subset T_m\mathcal{M}_{n,d}^{\text{sm}}$. Thus for any $m \in U$, ϕ sends $\text{Ker}(h_*)_m$ to $\text{Ker}(h_*)_{\phi(m)}$ isomorphically.

Let X be the holomorphic vector field on $\mathcal{M}_{n,d}^{\text{sm}}$ given by $X = \phi_*\xi$. By Theorem 4.5, we may write X in the form $X = h^*(f)\xi + X_\mu$, for a holomorphic function f and holomorphic 1-form μ on \mathcal{A} . We claim that f is non-vanishing. It suffices to show that $h^*(f)$ is non-vanishing on U , for if $h^*(f)$ has a zero then it vanishes on a codimension 1 subspace, which must therefore meet U . If $(h^*f)(\phi(m)) = 0$, where $m \in U$ then $\phi_*(\xi_m) = X_{\phi(m)} \in \text{Ker}(h_*)_{\phi(m)}$. But this is impossible, as ξ_m is not in $\text{Ker}(h_*)_m$. Thus f is non-vanishing.

Let ν be the unique solution to $\mathcal{L}_{\xi\mathcal{A}}(\nu) - \nu = -\mu/f$ guaranteed by Lemma 5.1. We then have:

$$\begin{aligned} (e^{X_\nu} \circ \phi)_* \xi &= (e^{X_\nu})_* \phi_* \xi \\ &= (e^{X_\nu})_* (h^*(f)\xi + X_\mu) \\ &= (e^{X_\nu})_* (h^*(f)\xi) + X_\mu \\ &= h^*(f) (e^{X_\nu})_* \xi + X_\mu \\ &= h^*(f)(\xi + X_{-\mu/f}) + X_\mu \\ &= h^*(f)\xi. \end{aligned}$$

In this computation we have used the fact that $(e^{X_\nu})_* X_\mu = X_\mu$, which follows from Equation (4.3) and $(e^{X_\nu})_* (h^*(f)\xi) = h^*(f) (e^{X_\nu})_* (\xi)$ which follows from the fact that $h_*(X_\nu) = 0$. What we have shown is that $\psi = e^{X_\nu} \circ \phi$ sends \mathbb{C}^* -orbits to \mathbb{C}^* -orbits. As in the proof of Proposition 4.7, we deduce that if $p \in \mathcal{M}_{n,d}$ is a point where \mathbb{C}^* acts freely then there is a $c \in \mathbb{C}^*$ such that for every $\lambda \in \mathbb{C}^*$ we have either $\psi(m_\lambda(p)) = m_{c\lambda}(\psi(p))$, or $\psi(m_\lambda(p)) = m_{c\lambda^{-1}}(\psi(p))$. Putting $\lambda = 1$, we see that we must have $c = 1$. Differentiating we find that $\psi_*(\xi_p) = \pm \xi_{\psi(p)}$. Since the orbits where \mathbb{C}^* acts freely are dense we conclude that $\psi_*(\xi) = \pm \xi$, hence $f = \pm 1$.

Suppose that $\psi_*(\xi) = -\xi$. Then $\psi : \mathcal{M}_{n,d} \rightarrow \mathcal{M}_{n,d}$ is an automorphism with the property that for all $\lambda \in \mathbb{C}^*$:

$$(5.2) \quad \psi \circ m_\lambda = m_{\lambda^{-1}} \circ \psi.$$

Let $p \in \mathcal{M}_{n,d}$ be any point with $h(p) \neq 0$. Then, since the Hitchin map is proper, the sequence $m_{1/k}(\psi(p))$, $k = 1, 2, \dots$ has a convergent subsequence. On the other hand the sequence $m_k(p)$ does not have a convergent subsequence, so neither does the sequence $\psi(m_k(p))$, since ψ is a homeomorphism. This contradicts (5.2), so we must have $\psi_*(\xi) = \xi$. This means that $\psi = e^{X_\nu} \circ \phi$ commutes with the \mathbb{C}^* -action. Uniqueness of ν follows from Corollary 4.10. \square

Proposition 5.3. *Let $\text{Aut}_{\mathbb{C}^*}(\mathcal{M}_{n,d})$ be the subgroup of $\text{Aut}(\mathcal{M}_{n,d})$ consisting of automorphisms of $\mathcal{M}_{n,d}$ that commute with the \mathbb{C}^* -action. We have that $\text{Vert}_0(\mathcal{M}_{n,d})$*

is a normal subgroup of $\text{Aut}(\mathcal{M}_{n,d})$ and that $\text{Aut}(\mathcal{M}_{n,d})$ is the semi-direct product:

$$\text{Aut}(\mathcal{M}_{n,d}) = \text{Aut}_{\mathbb{C}^*}(\mathcal{M}_{n,d}) \ltimes \text{Vert}_0(\mathcal{M}_{n,d}).$$

Proof. By Proposition 5.2, we need only show that $\text{Vert}_0(\mathcal{M}_{n,d})$ is a normal subgroup of $\text{Aut}(\mathcal{M}_{n,d})$. In fact it is enough to show that if $\phi \in \text{Aut}_{\mathbb{C}^*}(\mathcal{M}_{n,d})$ and $e^{X_\mu} \in \text{Vert}_0(\mathcal{M}_{n,d})$, then $\phi \circ e^{X_\mu} \circ \phi^{-1} \in \text{Vert}_0(\mathcal{M}_{n,d})$. Consider the 1-parameter subgroup $\phi \circ e^{tX_\mu} \circ \phi^{-1}$. Clearly this subgroup integrates the vector field $Y = \phi_*(X_\mu)$. Therefore it suffices to show that Y is a vertical vector field, that is $h_*(Y) = 0$. Arguing as in the proof of Proposition 5.2, we see that any automorphism ϕ sends vertical vector fields to vertical vector fields. In particular, Y is vertical and $\phi \circ e^{X_\mu} \circ \phi^{-1} = e^Y \in \text{Vert}_0(\mathcal{M}_{n,d})$. \square

Proposition 5.4. *Let α be a holomorphic endomorphism of $T^*SU_{n,d}^s$, where $SU_{n,d}^s \subseteq SU_{n,d}$ is the locus of stable bundles. Then α is a constant multiple of the identity.*

Proof. Let ϕ_s be the 1-parameter family of automorphisms of $T^*SU_{n,d}^s$, which acts fibrewise by $(e^{s\alpha^t})$. Here α^t is the endomorphism of the cotangent bundle induced by α . We claim that there are automorphisms $\psi_s : \mathcal{A} \rightarrow \mathcal{A}$ such that we have a commutative diagram

$$\begin{array}{ccc} T^*SU_{n,d}^s & \xrightarrow{\phi_s} & T^*SU_{n,d}^s \\ \downarrow h & & \downarrow h \\ \mathcal{A} & \xrightarrow{\psi_s} & \mathcal{A} \end{array}$$

and moreover the ψ_s commute with the \mathbb{C}^* -action on \mathcal{A} . Indeed this follows by an argument identical to the proof of [19, Proposition 2.1]. We have that ψ_s preserves the discriminant locus $\mathcal{D} \subset \mathcal{A}$, by [19, Proposition 2.2]. It follows that ϕ_s sends $T^*SU_{n,d}^s \cap \mathcal{M}_{n,d}^{\text{reg}}$ to itself. For any $b \in \mathcal{A}^{\text{reg}}$ we have that ϕ_s gives a birational isomorphism between $h^{-1}(b)$ and $h^{-1}(\psi_s(b))$. This is a birational isomorphism of abelian varieties and it follows that it extends to an isomorphism between $h^{-1}(b)$ and $h^{-1}(\psi_s(b))$. Thus ϕ_s extends as a 1-parameter family of automorphisms of $\mathcal{M}_{n,d}^{\text{reg}}$ (c.f., [19, Page 248]). By Theorem 4.5, we have that the vector field on $\mathcal{M}_{n,d}^{\text{reg}}$ generating the 1-parameter family ψ_s has the form $X = h^*(f)\xi + X_\mu$, where f is a holomorphic function on \mathcal{A}^{reg} and μ is a holomorphic 1-form on \mathcal{A}^{reg} . Moreover, ϕ_s is defined on $T^*SU_{n,d}^s$ and the restriction of the Hitchin map $h|_{T^*SU_{n,d}^s} : T^*SU_{n,d}^s \rightarrow \mathcal{A}$ is surjective [19, Lemma 1.4]. It follows that f and μ extend to the whole of \mathcal{A} . Then since ϕ_s commutes with the \mathbb{C}^* -action we can use Corollary 4.10 to deduce that $\mu = 0$ and f is constant. Thus, since X is a constant multiple of ξ we see that the automorphisms ϕ_s are given by the \mathbb{C}^* -action and hence α is a multiple of the identity. \square

Let $SU_{n,d}$ be the moduli space of rank n , degree d , semi-stable bundles with fixed determinant L_0 . Any automorphism $\phi : SU_{n,d} \rightarrow SU_{n,d}$ can be differentiated giving an automorphism $\phi_* = (\phi_*)^{-1} : T^*SU_{n,d}^{\text{sm}} \rightarrow T^*SU_{n,d}^{\text{sm}}$. It is clear from Theorem 1.1 that such automorphisms automatically extend to automorphisms of the Higgs bundle moduli space. Let $\text{Aut}(SU_{n,d})$ be the group of automorphisms of $SU_{n,d}$. We have just argued that $\text{Aut}(SU_{n,d})$ is in a natural way a subgroup of $\text{Aut}(\mathcal{M}_{n,d})$.

Theorem 5.5. *Let Σ have genus $g \geq 3$. We have an isomorphism $\text{Aut}_{\mathbb{C}^*}(\mathcal{M}_{n,d}) = \mathbb{C}^* \times \text{Aut}(\mathcal{SU}_{n,d})$, where the subgroup $\mathbb{C}^* \subset \text{Aut}_{\mathbb{C}^*}(\mathcal{M}_{n,d})$ is the usual \mathbb{C}^* -action on \mathcal{M} . Therefore, using Proposition 5.3, we have an isomorphism:*

$$\text{Aut}(\mathcal{M}_{n,d}) = (\mathbb{C}^* \times \text{Aut}(\mathcal{SU}_{n,d})) \ltimes \text{Vert}_0(\mathcal{M}_{n,d}).$$

Proof. It is clear that $\mathbb{C}^* \times \text{Aut}(\mathcal{SU}_{n,d}) \subseteq \text{Aut}_{\mathbb{C}^*}(\mathcal{M}_{n,d})$, so we only need to show the reverse inclusion $\text{Aut}_{\mathbb{C}^*}(\mathcal{M}_{n,d}) \subseteq \mathbb{C}^* \times \text{Aut}(\mathcal{SU}_{n,d})$. Let $\phi : \mathcal{M}_{n,d} \rightarrow \mathcal{M}_{n,d}$ be an automorphism of $\mathcal{M}_{n,d}$ which commutes with the \mathbb{C}^* -action. Let $U \subset \mathcal{M}_{n,d}$ be the open subset $U = T^*\mathcal{SU}_{n,d}^s \cap \phi(T^*\mathcal{SU}_{n,d}^s)$. Note that the complement $V = \mathcal{M}_{n,d} - T^*\mathcal{SU}_{n,d}^s$ is an analytic subset and that $U = \mathcal{M}_{n,d} - (V \cup \phi(V))$, so U is dense in $\mathcal{M}_{n,d}$. Let $W \subseteq \mathcal{SU}_{n,d}^s$ be the image of U under the projection $p : T^*\mathcal{SU}_{n,d}^s \rightarrow \mathcal{SU}_{n,d}^s$. Then W is an open subset since p is an open mapping and it is easy to see that W is dense in $\mathcal{SU}_{n,d}^s$, since $U \subseteq T^*\mathcal{SU}_{n,d}^s$ is dense in $T^*\mathcal{SU}_{n,d}^s$.

Let $E \in W$. By definition of W this means that there is a Higgs field $\Phi \in H^0(\Sigma, \text{End}_0(E) \otimes K)$, such that $(E, \Phi) \in U$. In turn, this means that E is a stable bundle and $\phi(E, \Phi) = (F, \Phi')$, where F is also stable. By \mathbb{C}^* -invariance it follows that $\phi(E, \lambda\Phi) = (F, \lambda\Phi')$ for all $\lambda \in \mathbb{C}^*$. Taking the limit as $\lambda \rightarrow 0$ and using continuity of ϕ , we get $\phi(E, 0) = (F, 0)$. If we think of W as a subset of $\mathcal{M}_{n,d}$ by the inclusions $W \subseteq \mathcal{SU}_{n,d} \subset \mathcal{M}_{n,d}$, then we have just shown that $\phi(W) \subseteq \mathcal{SU}_{n,d}$. Then since $\mathcal{SU}_{n,d}$ is closed in $\mathcal{M}_{n,d}$ and since W is dense in $\mathcal{SU}_{n,d}$, we have $\phi(\mathcal{SU}_{n,d}) \subseteq \mathcal{SU}_{n,d}$ by continuity. This shows that ϕ restricts to an automorphism of $\mathcal{SU}_{n,d}$, i.e., there exists $\psi \in \text{Aut}(\mathcal{SU}_{n,d})$ such that $\phi|_{\mathcal{SU}_{n,d}} = \psi$.

Let $(E, \Phi) \in T^*\mathcal{SU}_{n,d}^s$ and set $(F, \Phi') = \phi(E, \Phi)$. So by \mathbb{C}^* -equivariance, $\phi(E, \lambda\Phi) = (F, \lambda\Phi')$. In the limit as $\lambda \rightarrow 0$, we have by continuity of ϕ that $(F, \lambda\Phi') \rightarrow \phi(E, 0) = (\psi(E), 0) \in \mathcal{SU}_{n,d}^s$. Then since $T^*\mathcal{SU}_{n,d}^s$ is open in $\mathcal{M}_{n,d}$ we have that $(F, \lambda\Phi') \in T^*\mathcal{SU}_{n,d}^s$, for all small enough λ . Thus F is stable and $(F, \lambda\Phi')$ is in $T^*\mathcal{SU}_{n,d}^s$ for all λ . In particular setting $\lambda = 1$, we get that $(F, \Phi') = \phi(E, \Phi) \in T^*\mathcal{SU}_{n,d}^s$. This shows that ϕ restricts to an automorphism of $T^*\mathcal{SU}_{n,d}^s$. Our argument also shows that $p(\phi(m)) = \psi(p(m))$ for any $m \in T^*\mathcal{SU}_{n,d}^s$.

Let $\psi_* = (\psi^*)^{-1} : T^*\mathcal{SU}_{n,d}^s \rightarrow T^*\mathcal{SU}_{n,d}^s$ be the automorphism of $T^*\mathcal{SU}_{n,d}^s$ obtained by differentiating ψ . From Theorem 1.1, we see that ψ_* extends to an automorphism of $\mathcal{M}_{n,d}$ which commutes with the \mathbb{C}^* -action. Composing ϕ with $(\psi_*)^{-1}$, we reduce to the case that $\phi|_{\mathcal{SU}_{n,d}} = \text{id}$. So the restriction of ϕ to $T^*\mathcal{SU}_{n,d}^s$ acts as a fibre preserving automorphism covering the identity. Since ϕ commutes with the \mathbb{C}^* -action, ϕ descends to an automorphism of the projective cotangent bundle of $\mathcal{SU}_{n,d}^s$. This shows that ϕ acts linearly on the fibres of $T^*\mathcal{SU}_{n,d}^s$. It follows from Proposition 5.4, that such an automorphism is given by the \mathbb{C}^* -action and the theorem follows. \square

6. SUBGROUPS PRESERVING ADDITIONAL STRUCTURES

6.1. Hyper-Kähler geometry of the Higgs bundle moduli space. As we recall, the moduli space $\mathcal{M}_{n,d}$ carries a natural hyper-Kähler structure. To describe this we need to recall that $\mathcal{M}_{n,d}$ can also be viewed as the moduli space of solutions to the Hitchin equations. Let E be a fixed choice of a smooth, rank n degree d complex vector bundle, equip E with a Hermitian metric and let $L_0 = \det(E)$

with the induced metric. We let $\mathfrak{sl}(E) = \text{End}_0(E)$ be the bundle of trace-free endomorphisms of E and $\mathfrak{su}(E) \subset \mathfrak{sl}(E)$ the bundle of skew-adjoint trace-free endomorphisms of E . We let $\Omega^j(\Sigma, \mathfrak{sl}(E))$ denote the space of j -form valued sections of $\mathfrak{sl}(E)$. The adjoint map $A \mapsto A^*$ extends to an anti-linear involution $(\cdot)^* : \Omega^j(\Sigma, \mathfrak{sl}(E)) \rightarrow \Omega^j(\Sigma, \mathfrak{sl}(E))$ and the Hodge star $*$ extends to a linear map $*$: $\Omega^j(\Sigma, \mathfrak{sl}(E)) \rightarrow \Omega^{2-j}(\Sigma, \mathfrak{sl}(E))$.

The complex structure on Σ gives Σ an orientation. Let vol_Σ be a volume form on Σ inducing the same orientation and such that $\int_\Sigma \text{vol}_\Sigma = 1$. This determines a real valued L^2 -inner product on $\Omega^*(\Sigma, \mathfrak{sl}(E))$:

$$\langle \alpha, \beta \rangle = \frac{1}{2} \int_\Sigma \text{Tr}(\alpha^* \wedge * \beta) + \text{Tr}(\beta^* \wedge * \alpha).$$

If $\bar{\partial}_E$ is a $\bar{\partial}$ -operator on E , we let ∇_E denote the associated Chern connection, the unique unitary connection on E such that $(\nabla_E)^{0,1} = \bar{\partial}_E$ and we let $F_E \in \Omega^2(\Sigma, \mathfrak{su}_E)$ be the curvature of ∇_E . Fix a choice of hermitian connection ∇_{L_0} on L_0 with curvature $F_{L_0} = -2\pi i d \text{vol}_\Sigma$. If $\bar{\partial}_E$ is a holomorphic structure on E , we will say that $\det(E, \nabla_E) = (L_0, \nabla_{L_0})$ if the connection on L_0 induced by ∇_E equals ∇_{L_0} .

Let $(\bar{\partial}_E, \Phi)$ be a pair consisting of a $\bar{\partial}$ -operator $\bar{\partial}_E$ on E such that $\det(E, \nabla_E) \cong (L_0, \nabla_{L_0})$ and Φ a section of $\Omega^{1,0}(\Sigma, \mathfrak{sl}(E))$. The space of such pairs is an affine space modelled on $\Omega^{0,1}(\Sigma, \mathfrak{sl}(E)) \oplus \Omega^{1,0}(\Sigma, \mathfrak{sl}(E))$. The *Hitchin equations* for $(\bar{\partial}_E, \Phi)$ are:

$$\begin{aligned} F_E + [\Phi, \Phi^*] &= -2\pi i \mu(E) \text{vol}_\Sigma \otimes Id, \\ \bar{\partial}_E \Phi &= 0. \end{aligned}$$

Let $\mathcal{M}_{n, L_0, \nabla_{L_0}}^{\text{Hit}}$ denote the moduli space of solutions to the Hitchin equations modulo unitary gauge transformations (of rank n , with trace-free Higgs field and fixed determinant (L_0, ∇_{L_0})). Standard gauge-theoretic constructions give a topology on $\mathcal{M}_{n, L_0, \nabla_{L_0}}^{\text{Hit}}$. Observe that if (E, Φ) is a solution to the Hitchin equations, then (E, Φ) is a Higgs bundle. Moreover, it can be shown that (E, Φ) is *polystable*, i.e., a direct sum of stable Higgs bundles of the same slope. Since polystable Higgs bundles are semi-stable we have a natural map $\iota : \mathcal{M}_{n, L_0, \nabla_{L_0}}^{\text{Hit}} \rightarrow \mathcal{M}_{n, L_0}$. A theorem of Hitchin [14] and Simpson [26] establishes a Hitchin-Kobayashi type correspondence for Higgs bundles. This correspondence states that a Higgs bundle (E, Φ) is in the image of ι if and only if it is polystable. But every S -equivalence class of Higgs bundles has a unique polystable object, so ι is a bijection, in fact a homeomorphism.

Remark 6.1. The moduli space $\mathcal{M}_{n, L_0, \nabla_{L_0}}^{\text{Hit}}$ essentially depends on (L_0, ∇_{L_0}) only through the degree $d \bmod n$. To see this, let (L, ∇_L) be a line bundle of degree a and let ∇_L be a connection on L with curvature $F_L = -2\pi i a \text{vol}_\Sigma$. Tensoring solutions of the Hitchin equations by (L, ∇_L) produces a commutative square:

$$\begin{array}{ccc} \mathcal{M}_{n, L_0, \nabla_{L_0}}^{\text{Hit}} & \xrightarrow{\otimes (L, \nabla_L)} & \mathcal{M}_{n, L_0 \otimes L^n, \nabla_{L_0} \otimes Id + Id \otimes (\nabla_L)^{\otimes n}} \\ \downarrow \iota & & \downarrow \iota \\ \mathcal{M}_{n, L_0} & \xrightarrow{\otimes L} & \mathcal{M}_{n, L_0 \otimes L^n} \end{array}$$

In a similar manner, one can show that the choice of volume form vol_Σ is completely arbitrary. As such we may safely write $\mathcal{M}_{n,d}^{\text{Hit}}$ for the moduli space of solutions of the Hitchin equations and observe that we have a homeomorphism $\mathcal{M}_{n,d}^{\text{Hit}} \cong \mathcal{M}_{n,d}$.

One upshot of the identification $\mathcal{M}_{n,d} \cong \mathcal{M}_{n,d}^{\text{Hit}}$ is that $\mathcal{M}_{n,d}^{\text{Hit}}$ carries a natural hyper-Kähler structure. Indeed, $\mathcal{M}_{n,d}^{\text{Hit}}$ may be constructed as an infinite dimensional hyper-Kähler quotient [14]. Let $m = (\bar{\partial}_E, \Phi) \in \mathcal{M}_{n,d}^{\text{Hit}}$ be a smooth point. From the hyper-Kähler quotient construction, it follows that the tangent space $T_m \mathcal{M}_{n,d}^{\text{Hit}}$ can be described in terms of harmonic representatives. Under this identification the tangent space $T_m \mathcal{M}_{n,d}^{\text{Hit}}$ is given by pairs $(\dot{A}, \dot{\Phi}) \in \Omega^{0,1}(\Sigma, \mathfrak{sl}(E)) \oplus \Omega^{1,0}(\Sigma, \mathfrak{sl}(E))$ such that:

$$\begin{aligned}\bar{\partial}_E \dot{\Phi} + [\dot{A}, \Phi] &= 0, \\ \partial_E \dot{A} + [\dot{\Phi}, \Phi^*] &= 0,\end{aligned}$$

where ∂_E is the $(1,0)$ -part of the Chern connection ∇_E associated to $\bar{\partial}_E$. The hyper-Kähler structure on the smooth locus of $\mathcal{M}_{n,d}^{\text{Hit}}$ is given by a metric g and complex structures I, J, K satisfying the quaternionic relations $IJ = K$. In terms of harmonic representatives the metric g is given by:

$$g((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \frac{i}{2} \int_\Sigma \text{Tr}(\dot{A}_1^* \wedge \dot{A}_2 + \dot{A}_2^* \wedge \dot{A}_1) - \text{Tr}(\dot{\Phi}_1^* \wedge \dot{\Phi}_2 + \dot{\Phi}_2^* \wedge \dot{\Phi}_1),$$

and the complex structures I, J, K by:

$$I(\dot{A}, \dot{\Phi}) = (i\dot{A}, i\dot{\Phi}), \quad J(\dot{A}, \dot{\Phi}) = (i\dot{\Phi}^*, -i\dot{A}^*), \quad K(\dot{A}, \dot{\Phi}) = (-\dot{\Phi}^*, \dot{A}^*).$$

Note that I is just the natural complex structure on $\mathcal{M}_{n,d}$ as introduced in Section 2. Let $\omega_I, \omega_J, \omega_K$ denote the associated Kähler forms:

$$\omega_I(X, Y) = g(IX, Y), \quad \omega_J(X, Y) = g(JX, Y), \quad \omega_K(KX, Y) = g(KX, Y).$$

We also define complex 2-forms $\Omega_I, \Omega_J, \Omega_K$ by:

$$\Omega_I = \omega_J + i\omega_K, \quad \Omega_J = \omega_K + i\omega_I, \quad \Omega_K = \omega_I + i\omega_J.$$

Then Ω_I is a closed complex symplectic 2-form of type $(2,0)$ with respect to I and similarly for Ω_J, Ω_K . Note that this definition of Ω_I agrees with our previous definition, Equation (2.1).

6.2. Symmetry groups. Our goal in this section is to determine the subgroups given in Definition 1.3.

Lemma 6.2. *Suppose $g \geq 3$. Then $\mathcal{M}_{n,d}^{\text{sm}}$ is simply-connected.*

Proof. Let $\mathcal{SU}_{n,d}^s \subset \mathcal{SU}_{n,d}$ be the locus of stable bundles in $\mathcal{SU}_{n,d}$. We have that $\mathcal{SU}_{n,d}^s$ is simply-connected [1, 10]. It is also known that for $g \geq 3$, $\mathcal{SU}_{n,d}^s = \mathcal{SU}_{n,d}^{\text{sm}}$ [21]. In particular, it follows that $T^* \mathcal{SU}_{n,d}^{\text{sm}}$ is simply-connected. The lemma follows since the codimension of the complement of $T^* \mathcal{SU}_{n,d}^{\text{sm}} \subseteq \mathcal{M}_{n,d}^{\text{sm}}$ is at least 2. \square

Lemma 6.3. *Let μ be a holomorphic 1-form on \mathcal{A} and X_μ the corresponding holomorphic vector field on $\mathcal{M}_{n,d}$.*

- (1) *If X_μ preserves J (i.e., $\mathcal{L}_{X_\mu} J = 0$), then $\mu = 0$.*
- (2) *If X_μ preserves g (i.e., $\mathcal{L}_{X_\mu} g = 0$), then $\mu = 0$.*

Proof. We first show that condition (1) implies condition (2), that is, if X_μ preserves J then it also preserves g . To see this suppose that X_μ preserves J . Over $\mathcal{M}_{n,d}^{\text{reg}}$ we have an orthogonal decomposition $T\mathcal{M}_{n,d}^{\text{reg}} = \text{Ker}(h_*) \oplus J\text{Ker}(h_*)$. Let $V = \text{Ker}(h_*)$ and $H = JV$. Since X_μ preserves the subbundle V , it must also preserve H . It follows that $\mathcal{L}_{X_\mu}g$ is a section of $S^2(V^*) \oplus S^2(H^*)$. On the other hand observe that $\Omega_I(JX, Y) = \omega_J(JX, Y) + i\omega_K(JX, Y) = -g(X, Y) - i\omega_I(X, Y)$. So for any real vector fields X, Y we have $g(X, Y) = -\text{Re}(\Omega_I(JX, Y))$. However we also see that

$$\begin{aligned}\mathcal{L}_{X_\mu}\Omega_I &= di_{X_\mu}\Omega_I + i_{X_\mu}d\Omega_I \\ &= dh^*(\mu) = h^*(d\mu).\end{aligned}$$

So if X_μ preserves J , then for all X, Y we have:

$$\mathcal{L}_{X_\mu}g(X, Y) = -\text{Re}(h^*(d\mu)(JX, Y)).$$

However the right hand side vanishes whenever X and Y are either both vertical or both horizontal. So this equality is only possible if both sides vanish and hence X_μ preserves g .

Now suppose that X_μ preserves g . Clearly this implies that X_μ preserves ω_I , so $i_{X_\mu}\omega_I$ is a closed 1-form on $\mathcal{M}_{n,d}^{\text{sm}}$. By Lemma 6.2, $\mathcal{M}_{n,d}^{\text{sm}}$ is simply-connected so there is a smooth function g on $\mathcal{M}_{n,d}^{\text{sm}}$ such that $i_{X_\mu}\omega_I = dg$. On the other hand, given a non-singular fibre $h^{-1}(b) \subset \mathcal{M}_{n,d}^{\text{reg}}$, we have that ω_I restricts to a Kähler form on $h^{-1}(b)$ and that the flow of X_μ on $h^{-1}(b)$ is given by translations. As is well known, the action of a complex torus on itself by translation is not Hamiltonian. Hence we have a contradiction unless X_μ vanishes on $h^{-1}(b)$. Since $h^{-1}(b)$ was an arbitrary non-singular fibre this shows that $X_\mu = 0$ and hence $\mu = 0$. \square

Corollary 6.4. *Let $Y = f\xi + X_\mu$ be a holomorphic vector field on $\mathcal{M}_{n,d}^{\text{sm}}$, where $f \in \mathbb{C}$ is a constant and μ is a holomorphic 1-form on \mathcal{A} .*

- (1) *If Y preserves J , then $\mu = 0$ and $f \in \mathbb{R}$.*
- (2) *If Y preserves g , then $\mu = 0$ and $f \in i\mathbb{R}$.*

Proof. (1). First we note that ξ preserves J but does not preserve g , while $i\xi$ preserves g but does not preserve J . If Y preserves J then so does the commutator $[\xi, Y] = [\xi, f\xi + X_\mu] = X_\tau$, where $\tau = \mathcal{L}_{\xi\mathcal{A}}(\mu) - \mu$. Then by Lemma 6.3, we have $\tau = 0$, hence $\mu = 0$. So $Y = f\xi$ and for this to preserve J we must have that f is real. The proof of (2) follows by a similar argument. \square

For any holomorphic function f on \mathcal{A} , we have the corresponding Hamiltonian vector field X_f which can be integrated to a Hamiltonian flow e^{X_f} . Clearly the Hamiltonian flows define a subgroup of $\text{Vert}_0(\mathcal{M}_{n,d})$, which we will denote by $\text{Ham}(\mathcal{M}_{n,d})$. Then the map $\mathcal{O}(\mathcal{A}) \rightarrow \text{Ham}(\mathcal{M}_{n,d})$ sending a holomorphic function f to e^{X_f} is a surjective homomorphism with Kernel the constant functions on \mathcal{A} .

Theorem 6.5. *Under the isomorphism $\text{Aut}(\mathcal{M}_{n,d}) \cong (\mathbb{C}^* \times \text{Aut}(\mathcal{SU}_{n,d})) \ltimes \text{Vert}_0(\mathcal{M}_{n,d})$ of Theorem 5.5, the subgroups given in Definition 1.3 are as follows:*

- (1) $\text{Aut}_{\text{Sympl}}(\mathcal{M}_{n,d}) = (\{1\} \times \text{Aut}(\mathcal{SU}_{n,d})) \ltimes \text{Ham}(\mathcal{M}_{n,d})$,
- (2) $\text{Aut}_{\text{Isom}}(\mathcal{M}_{n,d}) = (U(1) \times \text{Aut}(\mathcal{SU}_{n,d}))$,
- (3) $\text{Aut}_Q(\mathcal{M}_{n,d}) = (\mathbb{R}_+ \times \text{Aut}(\mathcal{SU}_{n,d}))$,
- (4) $\text{Aut}_{HK}(\mathcal{M}_{n,d}) = (\{1\} \times \text{Aut}(\mathcal{SU}_{n,d}))$,

where $U(1) \subset \mathbb{C}^*$ is the subgroup of unit complex numbers and $\mathbb{R}_+ \subset \mathbb{C}^*$ is the subgroup of positive real numbers.

Proof. From the description of g, I, J, K given in Section 6.1, it is straightforward to see that the subgroup $\text{Aut}(\mathcal{SU}_{n,d})$ preserves the full hyper-Kähler structure. Therefore it suffices to only consider elements in $\mathbb{C}^* \ltimes \text{Vert}_0(\mathcal{M}_{n,d})$. Such an element can be written in the form $\phi = e^{X_\mu} \circ m_\lambda$, where μ is a holomorphic 1-form on \mathcal{A} and $\lambda \in \mathbb{C}^*$.

(1). For this note that $(\phi^{-1})^* \Omega_I = (e^{-X_\mu})^* m_{\lambda^{-1}}^* \Omega_I = (e^{-X_\mu})^* \lambda^{-1} \Omega_I = \lambda^{-1} (\Omega_I - h^*(d\mu))$. So ϕ preserves Ω_I if and only if $\Omega_I = \lambda^{-1} (\Omega_I - h^*(d\mu))$. Clearly this is possible if and only if $\lambda = 1$ and $d\mu = 0$. Therefore μ is a closed 1-form on \mathcal{A} and there exists a holomorphic function f such that $\mu = df$.

(2). Since $i\xi$ preserves g , it follows that so does $(e^{X_\mu} \circ m_\lambda)_* i\xi = (e^{X_\mu})_* m_{\lambda*} i\xi = (e^{X_\mu})_*(i\xi) = i(\xi + X_\tau)$, where $\tau = \mathcal{L}_{\xi^A}(\mu) - \mu$. From Corollary 6.4, it follows that $\tau = 0$ and therefore $\mu = 0$.

(3). Since ξ preserves J , it follows that so does $(e^{X_\mu} \circ m_\lambda)_* \xi = (e^{X_\mu})_* m_{\lambda*} \xi = (e^{X_\mu})_*(\xi) = (\xi + X_\tau)$, where $\tau = \mathcal{L}_{\xi^A}(\mu) - \mu$. Once again, it follows from Corollary 6.4, that $\mu = 0$.

(4). This follows immediately from cases (2) and (3). \square

7. ANTI-AUTOMORPHISMS AND THE FULL ISOMETRY GROUP

7.1. Anti-automorphisms. In this section we will make the simplifying assumption that n and d are coprime. It follows that all semi-stable Higgs bundles are stable and that $\mathcal{M}_{n,d}$ is a smooth hyper-Kähler manifold. Likewise all semi-stable bundles are stable and $\mathcal{SU}_{n,d}$ is a smooth projective variety. If X is a complex manifold with complex structure I then by an *anti-automorphism* of X we mean a diffeomorphism $\phi : X \rightarrow X$ such that $\phi_* \circ I = -I \circ \phi_*$.

Theorem 7.1. *Suppose that n and d are coprime. Then $\mathcal{M}_{n,d}$ admits an anti-automorphism if and only if Σ admits an anti-automorphism.*

Proof. First suppose that \mathcal{M}_{n,L_0} admits an anti-automorphism $f : \Sigma \rightarrow \Sigma$. Fix an underlying smooth bundle E of rank n , degree d and choose a Hermitian structure on E . As in Section 6, we identify \mathcal{M}_{n,L_0} with $\mathcal{M}_{n,L_0}^{\text{Hit}}$, the moduli space of solutions to the Hitchin equations. We can view L_0 as a line bundle with unitary connection. Then $f^*(L_0)$ is also a line bundle with unitary connection and in particular has a holomorphic structure. Bearing this in mind, we obtain an anti-holomorphic map $f^* : \mathcal{M}_{n,L_0} \rightarrow \mathcal{M}_{n,f^*(L_0)}$ which sends a solution $(\bar{\partial}_E, \Phi)$ of the Hitchin equations to $f^*(E, \Phi) = (f^*(\bar{\partial}_E), f^*(\Phi^*))$, where $\bar{\partial}_E$ is the $(1,0)$ -part of the Chern connection of $\bar{\partial}_E$ (cf. [2]). This corresponds to pullback under f of the connection $\nabla = \nabla_E + \Phi + \Phi^*$. Let L be a line bundle with connection such that $L^n \cong L_0 f^*(L_0)$. Such an L exists as $L_0 f^*(L_0)$ has degree zero. Next, consider the map $\delta_L : \mathcal{M}_{n,f^*(L_0)} \rightarrow \mathcal{M}_{n,f^*(L_0)L^n} = \mathcal{M}_{n,L_0}$, which is the holomorphic map $(E, \Phi) \mapsto (E^* \otimes L, \Phi^t \otimes Id)$. The composition $f_L = \delta_L \circ f^* : \mathcal{M}_{n,L_0} \rightarrow \mathcal{M}_{n,L_0}$ is then an anti-automorphism of \mathcal{M}_{n,L_0} .

To prove the converse, we introduce the following notation: fix a choice of volume form vol_Σ on Σ . If j is a complex structure on Σ inducing the same orientation as vol_Σ , we let $\mathcal{M}_{n,L_0}(j)$ denote the moduli space of rank n , trace-free Higgs bundles with determinant L_0 associated to (Σ, j) . Now let j be a given complex structure on Σ and suppose that $\phi : \mathcal{M}_{n,L_0}(j) \rightarrow \mathcal{M}_{n,L_0}(j)$ is an anti-automorphism. Choose an orientation reversing homeomorphism $g : \Sigma \rightarrow \Sigma$ and suppose vol_Σ is chosen such that $g^*(vol_\Sigma) = -vol_\Sigma$. Note that such pairs (g, vol_Σ) certainly exist, as we could take g to be an involution. Further, choose a unitary line bundle with connection L , such that $L^n \cong L_0 g^*(L_0)$. Then $-g^*(j)$ is a complex structure on Σ inducing the same orientation as j . We have an anti-automorphism $g^* : \mathcal{M}_{n,L_0}(j) \rightarrow \mathcal{M}_{n,g^*(L_0)}(-g^*(j))$ given by $g^*(\bar{\partial}_E, \Phi) = (g^*(\bar{\partial}_E), g^*(\Phi^*))$. As above, consider the map $\delta_L : \mathcal{M}_{n,g^*(L_0)} \rightarrow \mathcal{M}_{n,g^*(L_0^*)L^n} = \mathcal{M}_{n,L_0}$, which is the holomorphic map $(E, \Phi) \mapsto (E^* \otimes L, \Phi^t \otimes Id)$. The composition $\delta_L \circ g^* \circ \phi : \mathcal{M}_{n,L_0}(j) \rightarrow \mathcal{M}_{n,L_0}(-g^*(j))$ is then an isomorphism of complex manifolds. Now we use the Torelli theorem for Higgs bundle moduli spaces [5, Theorem 1.1] to conclude that (Σ, j) and $(\Sigma, -g^*(j))$ are isomorphic. This means that there is a diffeomorphism $f : \Sigma \rightarrow \Sigma$ such that $f^*(-g^*(j)) = j$. Thus, $g \circ f : \Sigma \rightarrow \Sigma$ is an anti-automorphism of (Σ, j) . \square

7.2. The isometry group. In this section we will determine the isometry group $Isom(\mathcal{M}_{n,d})$ of $\mathcal{M}_{n,d}$.

Lemma 7.2. *Suppose n and d are coprime. The only covariantly constant endomorphisms of $TSU_{n,d}$ are linear combinations of I and the identity.*

Proof. Any endomorphism $E : TSU_{n,d} \rightarrow TSU_{n,d}$ can be uniquely written as a sum $E = A + B$, where $AI = IA$ and $BI = -IB$, namely $A = \frac{1}{2}(E - IEI)$, $B = \frac{1}{2}(E + IEI)$. If E is covariantly constant then so are A and B . In particular A corresponds to a holomorphic endomorphism of $T^{(1,0)}SU_{n,d}$. By Proposition 5.4, any such endomorphism is of the form λId , $\lambda \in \mathbb{C}$. This corresponds to $A = aId + bI$, where $\lambda = a + ib$, $a, b \in \mathbb{R}$. To finish the lemma it remains to show that there are no constant endomorphisms B with $BI = -IB$. Such an endomorphism corresponds to an anti-linear map $B : T^{(1,0)}SU_{n,d} \rightarrow T^{(0,1)}SU_{n,d}$. The hermitian metric on $SU_{n,d}$ defines an anti-linear isomorphism $h : T^{(0,1)}SU_{n,d} \rightarrow (T^{(1,0)}SU_{n,d})^*$. Thus the composition $h \circ B$ is a \mathbb{C} -linear covariantly constant endomorphism $h \circ B : T^{(1,0)}SU_{n,d} \rightarrow (T^{(1,0)}SU_{n,d})^*$.

If $h \circ B$ is an isomorphism then by taking determinants we obtain a trivialisation of the square of the canonical bundle. This contradicts the fact that the anti-canonical bundle is ample [25]. Therefore $h \circ B$ has a non-trivial kernel U , which has constant rank as $h \circ B$ is covariantly constant. Using h we get an orthogonal decomposition $T^{(1,0)}SU_{n,d} = U \oplus V$ which is preserved by the Levi-Civita connection. However if this decomposition is non-trivial we would obtain holomorphic endomorphisms of $T^{(1,0)}SU_{n,d}$ other than multiples of the identity. Since this can not happen, $V = 0$, $U = T^{(1,0)}SU_{n,d}$ and hence $B = 0$ as claimed. \square

Proposition 7.3. *Suppose n and d are coprime. Then any covariantly constant endomorphism of $T\mathcal{M}_{n,d}$ is a linear combination of I, J, K and the identity.*

Proof. Consider the involution $\iota : \mathcal{M}_{n,d} \rightarrow \mathcal{M}_{n,d}$ given by $\iota(E, \Phi) = (E, -\Phi)$. This is an isometry of $\mathcal{M}_{n,d}$ and is anti-symplectic in the sense that $\iota^*\Omega_I = -\Omega_I$. It

follows that the fixed point set of ι is a complex Lagrangian submanifold. If $\gamma(t)$ is a geodesic in $\mathcal{M}_{n,d}$ such that $\gamma'(0)$ is tangent to the fix point set, then the same is true of $\iota(\gamma(t))$. Uniqueness of geodesics then gives $\gamma(t) = \iota(\gamma(t))$ and hence the fixed point set of ι is totally geodesic. Clearly $SU_{n,d}$ is in the fixed point set of ι and since it is a closed submanifold of half the dimension on $\mathcal{M}_{n,d}$ it is actually a component of the fixed point set of ι . This shows in particular that $SU_{n,d} \subset \mathcal{M}_{n,d}$ is totally geodesic.

Let $\nabla_{\mathcal{M}}$ denote the Levi-Civita connection of $\mathcal{M}_{n,d}$ and ∇_{SU} the Levi-Civita connection of $SU_{n,d}$ with the induced metric. Since $SU_{n,d} \subset \mathcal{M}_{n,d}$ is totally geodesic, we have that $\nabla_{\mathcal{M}}|_{SU_{n,d}}$ respects the orthogonal decomposition $T\mathcal{M}_{n,d}|_{SU_{n,d}} = TSU_{n,d} \oplus NSU_{n,d}$, where $NSU_{n,d}$ is the normal bundle. We also have that the restriction of $\nabla_{\mathcal{M}}|_{SU_{n,d}}$ to the sub-bundle $TSU_{n,d}$ coincides with ∇_{SU} . Moreover, the complex structure J gives a covariantly constant isomorphism $J : TSU_{n,d} \rightarrow NSU_{n,d}$. Using this isomorphism we have an isomorphism of bundles with connections:

$$(7.1) \quad (T\mathcal{M}_{n,d}|_{SU_{n,d}}, \mathcal{M}|_{SU_{n,d}}) = (TSU_{n,d}, \nabla_{SU}) \oplus (TSU_{n,d}, \nabla_{SU}).$$

Suppose that ϕ is a covariantly constant endomorphism of $T\mathcal{M}_{n,d}$. The restriction of ϕ to $SU_{n,d}$ decomposes under (7.1) into

$$(7.2) \quad \phi|_{SU_{n,d}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C, D are covariantly constant endomorphisms of $TSU_{n,d}$. By Lemma 7.2, we have that A, B, C, D are linear combinations of $I|_{SU_{n,d}}$ and the identity. Let \mathcal{C} be the space of all covariantly constant endomorphisms of $T\mathcal{M}_{n,d}$. We have just shown that \mathcal{C} has real dimension at most 8. On the other hand, as I, J, K are covariantly constant, we have that \mathcal{C} is a non-trivial module over the quaternions. So the dimension of \mathcal{C} is either 4 or 8. If the dimension is 4 we have proven the proposition, so suppose that \mathcal{C} is 8-dimensional. This means that for any covariantly constant endomorphisms A, B, C, D of $TSU_{n,d}$, there is a covariantly constant endomorphism $\phi \in \mathcal{C}$ satisfying Equation (7.2).

Consider the case $A = Id$, $B = C = 0$, $D = -Id$. The corresponding $\phi \in \mathcal{C}$ is trace-free and satisfies $\phi^2 = Id$. Let $\mathcal{T}_+, \mathcal{T}_-$ be the ± 1 -eigenbundles of ϕ . Thus $T\mathcal{M}_{n,d} = \mathcal{T}_+ \oplus \mathcal{T}_-$ and $\mathcal{T}_+|_{SU_{n,d}} = TSU_{n,d}$, $\mathcal{T}_-|_{SU_{n,d}} = NSU_{n,d}$. This implies that locally $\mathcal{M}_{n,d}$ is isometric to a product. Moreover, around any point $m \in SU_{n,d}$ we also have that $\mathcal{M}_{n,d}$ is isometric to a product $U_m \times V_m$ of a neighborhood U_m of m in $SU_{n,d}$ with another space V_m . Since $\mathcal{M}_{n,d}$ is hyper-Kähler, it is Ricci flat and hence this implies U_m, V_m are both Ricci flat. As the point m was arbitrary, this would imply that $SU_{n,d}$ with its natural metric is Ricci flat, but this is impossible, since the anti-canonical bundle is ample [25]. \square

Corollary 7.4. *Suppose n and d are coprime. The Riemannian holonomy group of $\mathcal{M}_{n,d}$ is $Sp(m)$, $m = (n^2 - 1)(g - 1)$.*

Proof. Let $G \subseteq Sp(m)$ be the Riemannian holonomy group of $\mathcal{M}_{n,d}$ and $G^0 \subseteq G$ the identity component. Then G^0 is a closed Lie subgroup of $Sp(m)$ [6]. If G^0 is a proper subgroup then it must act reducibly on $T\mathcal{M}_{n,d}$. But this would contradict Proposition 7.3, so in fact $G^0 = G = Sp(m)$. \square

Theorem 7.5. *Suppose n and d are coprime.*

- *If Σ does not admit an anti-automorphism, then every isometry of $\mathcal{M}_{n,d}$ preserves I . Therefore $\text{Isom}(\mathcal{M}_{n,d}) = \text{Aut}_{\text{Isom}}(\mathcal{M}_{n,d}) \cong (U(1) \times \text{Aut}(\mathcal{SU}_{n,d}))$.*
- *If Σ admits an anti-automorphism then the subgroup of isometries of $\mathcal{M}_{n,d}$ preserving I has index 2 in the isometry group of $\mathcal{M}_{n,d}$.*

Proof. Let $\phi : \mathcal{M}_{n,d} \rightarrow \mathcal{M}_{n,d}$ be an isometry. Then $\phi^*(I)$ is a covariantly constant complex structure, hence by Proposition 7.3, $\phi^*(I)$ belongs to the 2-sphere of complex structures $\{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$. However, it is known that I is not isomorphic to any complex structure in this 2-sphere other than itself and $-I$ [14]. It follows that either $\phi^*(I) = I$ or $-I$. If there exists an isometry ϕ for which $\phi^*(I) = -I$ then by Theorem 7.1, Σ admits an anti-automorphism. Conversely if Σ admits an anti-automorphism $f : \Sigma \rightarrow \Sigma$ then, as in the proof of Theorem 7.1 we constructed an anti-automorphism \hat{f}_L of $\mathcal{M}_{n,d}$. It is easy to see that this is an isometry. \square

Remark 7.6. In the case that Σ admits an anti-automorphism $f : \Sigma \rightarrow \Sigma$ we can be more precise about the isometry group of $\mathcal{M}_{n,d}$. Recall as in the proof of Theorem 7.1, we constructed an anti-automorphism $\hat{f}_L : \mathcal{M}_{n,d} \rightarrow \mathcal{M}_{n,d}$ which is also an isometry. Then

$$\text{Isom}(\mathcal{M}_{n,d}) = (U(1) \times \text{Aut}(\mathcal{SU}_{n,L_d})) \cup \hat{f}_L \circ (U(1) \times \text{Aut}(\mathcal{SU}_{n,L_d})).$$

To completely describe the group structure of $\text{Isom}(\mathcal{M}_{n,d})$ one just needs to know (1) the element $\hat{f}_L^2 \in (U(1) \times \text{Aut}(\mathcal{SU}_{n,d}))$ and (2) the adjoint action of \hat{f}_L on the subgroup $U(1) \times \text{Aut}(\mathcal{SU}_{n,d})$. For (1) let $\sigma : \Sigma \rightarrow \Sigma$ be the automorphism $\sigma = f^2$ and set $M = L \otimes f^*(L^*)$. Then it is easy to see that $M^n \cong L_0 \otimes \sigma^*(L_0^*)$ and \hat{f}_L^2 is the automorphism of $\mathcal{M}_{n,d}$ given by $(E, \Phi) \mapsto (\sigma^*(E) \otimes M, \sigma^*(\Phi) \otimes \text{Id})$. For (2) we find that the adjoint action of \hat{f}_L on the factor $U(1)$ is complex conjugation. The adjoint action of \hat{f}_L on $\text{Aut}(\mathcal{SU}_{n,d})$ can be easily determined from the description of $\text{Aut}(\mathcal{SU}_{n,d})$ given in Theorem 1.1.

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